WYDZIAŁ MATEMATYKI
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Calabi-Yau threefolds with triple points and type III contractions

Advisor:
dr hab. Grzegorz Kapustka
Jagiellonian University

Candidate:
Kacper Grzelakowski
album no. 5694

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1. Introduction

By a Calabi-Yau threefold we mean a complex projective threefold X with $K_X = 0$ and $h^1(\mathscr{O}_X) = h^2(\mathscr{O}_X) = 0$. We require X to be normal yet we allow it to have some singularities, namely ordinary double and ordinary triple points. A primitive contraction is a birational morphism between Calabi-Yau threefolds decreasing the Picard rank by one. A contraction is of type I if it contracts a finite set of curves to points, of type II if it contracts a divisor to a point and of type III if it contracts a divisor to a curve. It is known that certain deformation classes of Calabi-Yau threefolds are linked with each other by so called extremal transitions, that is primitive contractions followed by a smoothing. These transitions are called conifold if the singularities in the image of the contraction are ordinary double points. In this thesis we aim to discuss the process of obtaining Calabi-Yau threefolds through a conifold transition involving a type III primitive contraction. This procedure can be easily generalized to complete intersection Calabi-Yau threefolds or Calabi-Yau threefolds in weighted projective spaces giving rise to even more examples.

There are two main problems we are interested in. First is the construction of new Calabi-Yau threefolds. We manage to obtain a family of Calabi-Yau threefolds containing a cone which give rise to Calabi-Yau threefolds of Picard ranks 3 and 2. We believe some of our examples are new as we have not located CY3's with the same Hodge numbers as obtained by us in the databases [43] and [44]. The second problem is the following. What is the possible genus of a curve $C \subset Y$ that is the image of a divisor $E \subset X$ through the type III contraction $\rho : X \to Y$? The highest genus we managed to obtain is 31 (see 6.10). It remains an open problem if there is a bound to this genus.

Our results are in the spirit of the so-called Reid's fantasy. M. Reid in [33] has conjectured that there could exist an irreducible space parametrizing non-algebraic Calabi-Yau threefolds such that any Calabi Yau threefold would be a small resolution of a degeneration of this family to something with ordinary double points. We can think of this space as a graph with each node representing one deformation class of a family of algebraic Calabi-Yau threefolds with edges between these nodes giving extremal transitions. Namely let $\mathcal{M}_1$ and

$\mathcal{M}_2$ be two deformation classes of Calabi-Yau threefolds. We have an arrow $\mathcal{M}_1 \to \mathcal{M}_2$ if for a general element $\tilde{X} \in \mathcal{M}_1$ there is a birational contraction morphism $\pi : \tilde{X} \to Y$ and a flat family $\mathcal{Y} \to (\Delta, 0)$ such that $\mathcal{Y}_0 \cong Y$ and $\mathcal{Y}_t \in \mathcal{M}_2$ for general $t \in \Delta$. Finding these edges is an actively studied problem. Wilson in [42] provided a detailed description of the Kähler cones of Calabi-Yau threefolds, showing in particular that codimension one faces of their cones correspond to primitive contractions. In [17] and [18] Gross described the conditions for the primitive Calabi-Yau threefolds obtained by such contraction to be smoothable thus giving a link between nodes of the graph.

It is known that for a smooth X a type III contraction $\rho : X \to Y$ of a divisor E to a smooth curve C deforms to a type I contraction providing $g(C) > 1$ [17, Thorem 1.3]. It is conjectured [39, Conjecture 6.7] that a sufficiently general transition arising from the type III contraction deforms to a conifold transition and we show in Corollary 2.21 that in the case of surfaces ruled over smooth curves of genus > 1 this conjecture holds.

We provide the formula describing the change in the Hodge numbers of threefolds under this transition. Namely we show

Proposition 1.1. *Let X be a smooth Calabi-Yau threefold containing a smooth surface E ruled over a smooth curve C of genus $g(C) > 1$, let $\rho : X \to Y$ be the primitive type III contraction of E and let $\tilde{Y}$ be the smooth Calabi-Yau obtained by deforming Y. Then*

(1) $h^{1,1}(\tilde{Y}) = h^{1,1}(X) - 1$
(2) $h^{1,2}(\tilde{Y}) = h^{1,2}(X) + 2p_a(C) - 3.$

To provide examples of type III smoothable contractions we study the geometry of Calabi-Yau threefolds containing a cone over a curve. We choose quintic threefolds, complete intersection of a quadric and a quartic in $\mathbb{P}^5$, complete intersection of two cubics in $\mathbb{P}^5$ and a sextic hypersurface in weighted projective space $\mathbb{P}(1:1:1:1:2)$ as the source of examples. Quintic threefolds are natural objects being hypersurfaces of a relatively low degree yet having interesting properties. The geometry of quintic threefolds has been studied for example in [1] or [31]. Nodal quintic threefolds have been analysed by Friedman [14] and van Straten [41], while the problem of the possible number of triple points on such a hypersurface has been the subject of a recent work by Kloosterman and Rams [30]. We hope that our results may find applications in solving that last problem.

We resolve singularities of threefolds with triple point at the vertex of a cone to obtain smooth threefolds containing a ruled surface and use type III contractions to construct new Calabi-Yau threefolds with Picard rank 2. The procedure we describe is illustrated on the diagram below. Here $\bar{X}$ is a threefold containing a cone over a smooth curve with a triple point at the vertex of the cone and nodes on its surface, $\tilde{X}$ is the threefold obtained by blowing up the triple point of $\bar{X}$ and X is the one obtained by the small resolution of nodes of $\tilde{X}$, Y is the image of the type III contraction of X and $\tilde{Y}$ is a smooth deformation of Y.

$$\begin{array}{ccccccc} & & & X & & & \\ & & \swarrow & & \searrow^{\rho} & & \\ \bar{X} & \longleftarrow & \tilde{X} & & Y & \longrightarrow & \tilde{Y} \in \mathscr{Y} \end{array}$$

We provide a generalization of the results of Cynk from [4] and [5] regarding the Hodge numbers of resolutions of singularities of hypersurfaces in $\mathbb{P}^4$ with ordinary double and triple points (Theorem 5.5) and use it to calculate the Hodge numbers of the threefolds obtained from quintics in question. In a similar manner we adapt the formula to calculate the Hodge numbers of threefolds obtained from the resolution of complete intersection threefolds containing ordinary double and triple points as the only singularities (Theorem 5.14). In our analysis we complete the results of Kapustka and Kapustka from [23] and [24] where they discuss primitive type II contractions of Calabi-Yau threefolds, describe their images and smoothing families and give the formula for calculating the Hodge numbers of the threefolds obtained through this process.

We also apply the formulas we have obtained to calculate the Hodge numbers of resolutions of certain Calabi-Yau threefolds admitting ordinary triple points and not containing a cone but having huge defect. We present these results in Table 9.2.

The last part of the thesis is dedicated to the discussion of the bounds pertaining the number of ordinary triple points on Calabi-Yau threefolds. Our interest was sparked by the aforementioned results of Kloosterman and Rams regarding the possible number of triple points on a Calabi-Yau quintic threefold in $\mathbb{P}^4$ [30]. They have proven that a quintic threefold with a reducible hyperplane section cannot have more that 10 ordinary triple points as the only singularities and constructed the example where this limit is reached. The question whether or not a quintic threefold can have 11 ordinary triple points remains open, it is known that the number 12 is impossible. There already exists an extensive literature on the subject of the number of double points on Calabi-Yau threefolds for example [20][23][41] yet the exact bounds are still difficult to obtain. For instance in the case of quintic threefold the best result is given by van Straten in [41] with 130 double points and it is not known whether or not it is indeed the upper bound. The case of ordinary triple points should be in general simpler, yet is much less disussed. We analyse the bound for complete intersection threefolds $X_{2,4}$, $X_{3,3} \subset \mathbb{P}^5$ and for the sextic hypersurface in $\mathbb{P}(1:1:1:1:2)$; the results are summarised in table 15.

All calculations were performed using the Macaulay2 software [16]. Numerical results are presented in the tables at the end of the paper. Parts of this thesis have been published as [21].

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2. Preliminary notions

Throughout the thesis we always work over the field of complex numbers $\mathbb{C}$. We work in either the projective space $\mathbb{P}^n$ or in a weighted projective space $\mathbb{P}(1:1:1:1:2)$. By an algebrac variety we mean an integral reduced scheme of a finite type.

Definition 2.1. A Calabi-Yau threefold is a projective complex variety X of dimension 3, possibly singular, with $K_X = 0$ and $h^1(\mathscr{O}_X) = 0$.

Definition 2.2. A primitive contraction of a Calabi-Yau threefold X is a birational morphism $f : X \to Y$ to a variety Y such that $\rho(Y) = \rho(X) - 1$ where $\rho(X)$ is the rank of the Picard group of X. A contraction is of type:

- I if it contracts a finite set of curves to points
- II if it contracts a divisor to a point
- III if it contracts a divisor to a curve.

We recall [39, Definition 1.4].

Definition 2.3. A process $T(X, Y, \tilde{Y})$ of going from X to $\tilde{Y}$, where $\pi : X \to Y$ is a birational contraction of a Calabi-Yau X onto a normal variety Y and $\tilde{Y}$ is a smoothing of Y, is called a geometric transition. A transition is called conifold if Y admits only ordinary double points as singularities.

Definition 2.4. A non-singular Calabi-Yau threefold X is primitive if there is no birational contraction $X \to Y$ with Y smoothable to a Calabi-Yau threefold which is not deformation equivalent to X.

2.1. **Kähler cone of Calabi-Yau threefolds.**

Definition 2.5. Let X be a normal variety over a field K of dimension d. We have:

- $Z_{d-1}(X)$ - the group of Weil divisors of X that is the free abelian group generated by prime divisors on X
- $Div(X)$ - the group of Cartier divisors of X that is $H^0(X, Rat(X)^\times/\mathscr{O}_X^\times)$ where $Rat(X)^\times$ is the sheaf of nonzero rational functions on X
- $Pic(X)$ - the group of line bundles on X
- $Z_1(X)$ - the free abelian group generated by reduced irreducible curves on X.

Denote by $\equiv$ the relation of numerical equivalence on both $Z_1(X)$ and $Pic(X)$. Then we additionally have:

- $N^1(X)_{\mathbb{R}} := \{Pic(X)/\equiv\} \otimes \mathbb{R}$
- $N_1(X)_{\mathbb{R}} := \{Z_1(X)/\equiv\} \otimes \mathbb{R}$
- $\bar{NE}(X)$ - the closed covex cone in $N_1(X)_{\mathbb{R}}$ generated by reduced irreducible curves on X.

Definition 2.6. We say that a divisor $L \in N^1(X)_{\mathbb{R}}$ is nef if $L \geq 0$ on $\bar{NE}(X)$.

Definition 2.7. We say that a line bundle $\mathscr{L}$ on X is very ample if it is basepoint-free and the associated morphism $f_{\mathscr{L}} : X \to \mathbb{P}^n$ is the closed immersion. We say that $\mathscr{L}$ is ample if for some $r > 0$ its tensor power $\mathscr{L}^{\otimes r}$ is very ample.

Definition 2.8. We say that a divisor $L \in N^1(X)_{\mathbb{R}}$ is ample if the corresponding line bundle $\mathscr{L} := \mathscr{O}(L)$ is ample.

Theorem 2.9 (Kleiman's criterion for ampleness)**.** *A divisor $L \in Pic(X)$ is ample if and only if the numerical class $L \in N^1(X)_{\mathbb{R}}$ gives a positive function on $\bar{NE}(X) - \{0\}$.*

Definition 2.10. We define the following subsets of $N^1(X)_{\mathbb{R}}$:

- $\mathrm{Amp}(X)$ the ample cone of X

- $\mathrm{Nef}(X)$ the nef cone of X

as the convex cones of ample (respectively nef) divisors of X.

By Kleiman's Theorem we can consider the nef cone as the closure of the ample cone.

Definition 2.11. Let X be a compact Kähler manifold. The Kähler cone $K(X) \subset H^{1,1}(X) \cap H^2(X, \mathbb{R})$ is the cone of Kähler classes of X.

Under the inclusion $N^1(X)_{\mathbb{R}} \subset H^{1,1}(X, \mathbb{R})$, we have $Amp(X) = K(X) \cap N^1_{\mathbb{R}}(X)$. For simply connected Calabi-Yau threefolds, since $h^1(\mathscr{O}_X) = h^2(\mathscr{O}_X) = 0$, the map $c_1 : Pic(X) \to H^2(X, \mathbb{Z})$ is an isomorphism. Thus $N^1(X)_{\mathbb{R}} \to H^{1,1}(X, \mathbb{R})$ is surjective and so $Amp(X) = K(X)$. It follows that $Nef(X) = \bar{K}(X)$.

Definition 2.12. Let X be a Calabi-Yau threefold. We define the cubic cone of X as $W = \{T \in Pic(X) \otimes \mathbb{R} : T^3 = 0\}$.

We have [42, Fact 1]:

Fact 2.13. *The cone $\bar{K}(X)$ is locally rational polyhedral away from W, the codimension one faces corresponding to primitive birational contractions on X.*

To reiterate, we can think of elements of the cone $K(X)$ as of the ample divisors on a Calabi-Yau threefold X and the elements of its closure $\bar{K}(X)$ as of nef divisors on X. Thus, by the above fact, nef divisors lying on a codimension one faces of the Kähler cone not in W provide primitive birational contractions of X. By that we mean that if D is such a divisor then the complete linear system $|mD|$ for some $m \in \mathbb{N}^{>0}$ gives rise to a morphism $\Phi_{|mD|} : X \to Y$ which is a primitive birational contraction.

2.2. **Deformation theory and type III contractions.** We use [17] and [18] as the reference.

When we perform a birational contraction of a Calabi-Yau threefold X we obtain a threefold Y with at worst canonical singularities. We are not always able to find a smoothing of such threefold. For Y having canonical singularities $Def(Y)$ can still be singular. It turns out that the obstructions to deforming Y are the obstructions to deforming a germ of the singularities of Y. In most cases though, we are able to control the codimension of the singular locus of $Def(Y)$ and this allows us to discuss the locus where it is smooth, thus providing a smoothing of a contracted manifold. For details we refer to [18, Section 2].

We are interested in the case when the contraction is of type III yet it turns out it is closely linked with the case of contractions of type I as will be shown below. We recall facts about deformations with a view towards the existence of a smoothing of a type III contraction. In particular we quote results describing when the type I contraction is smoothable and discuss how the type III contraction deforms to a type I contraction.

First we consider the deformation space of the resolution X of Y and recall when it is smooth. This will be useful when dicussing the deformation of a type III contraction when the curve in the image has $g = 1$. Let $(Y, 0)$ be a germ of an isolated rational complex threefold singularity. Let $\pi : (X, 0) \to (Y, 0)$ be the resolution of singularities. There is a natural map of germs of analytic spaces $Def(X) \to Def(Y)$ as $H^1(\mathscr{O}_X) = 0$. We denote by $\mathscr{O}_{Y,0}$ the local ring of Y at the origin with maximal ideal $\mathfrak{m}$. Let T^1 be the tangent space of $Def(Y)$. We quote the following results without proofs.

Lemma 2.14. [18, Lemma 3.2] *The tangent space to $Def(X)$ is $H^0(R^1\pi_*\mathcal{T}_X)$. The tangent space of $Def(Y)$ is $T^1 = H^2_Z(\mathcal{T}_Y)$ where $\mathcal{T}_Y = Hom_{\mathscr{O}_Y}(\Omega^1_Y, \mathscr{O}_Y)$, and we have an exact sequence of $\mathscr{O}_{Y,0}$-modules*

$$H^0(R^1\pi_*\mathcal{T}_X) \to H^2_Z(\mathcal{T}_Y) \to T' \to 0$$

with $T' = Ker(H^2_E(\mathcal{T}_X) \to H^0(R^2\pi_*\mathcal{T}_X))$, $Z = Sing(Y)$ and E the exceptional locus of π. The map $H^0(R^1\pi_*\mathcal{T}_X) \to H^2_Z(\mathcal{T}_Y)$ is the differential of the map $Def(X) \to Def(Y)$.

Proposition 2.15. [18, Proposition 3.4] *Let $X \to Y$ be a crepant resolution of an isolated rational Gorenstein threefold singularity $(Y, 0)$. Then $Def(X)$ is non-singular.*

We move on to discuss the conditions under which the image of the contraction is smoothable. We have the following results:

Theorem 2.16. *Let X be a non-singular Calabi-Yau threefold, and $\pi : X \to Y$ be a birational contraction morphism such that Y has isolated, canonical, complete intersection singularities. Then there is a deformation of Y which smooths all singular points of Y except possibly the ordinary double points of Y. In particular, if Y has no ordinary double points, then Y is smoothable.*

Theorem 2.17. *Suppose $\pi : X \to Y$ is a primitive type I contraction. Then Y is smoothable unless π is the contraction of a single $\mathbb{P}^1$ to an ordinary double point.*

The above result will be particulary usefull as it shows that in case the type III contraction deforms to a type I contraction with more than one $\mathbb{P}^1$ being contracted the image threefold will be smoothable. We have the following theorems:

Theorem 2.18. [18, Theorem 1.1] *Let $\pi : X \to Y$ be a primitive type III contraction of a non-singular Calabi-Yau threefold X, contracting an exceptional divisor E to a curve C. Then:*

(1) C is a non-singular curve
(2) $\pi : E \to C$ is a conic bundle over C, and each fibre is either a non-singular conic, a union of two lines meeting at a point or a doubled line. If the general fibre is a non-singular curve, then E is normal. In this case, the singularities which appear on E are A_n $(n \geq 0)$ singulariteis at the singular point of a reducible reduced fibre, or two A_1 singularities on a non-reduced fibre.

Theorem 2.19. [18, Theorem 1.2] *Let $\pi : X \to Y$ be a primitive type III contraction of a non-singular Calabi-Yau threefold X contracting a divisor E to a curve C. Let $\tilde{E}$ be the* *normalization of E, $f : \tilde{E} \to X$ the induced map, and $\tilde{E} \to \tilde{C} \to C$ the Stein factorizarion. Then the image of the natural map $Def(f) \to Def(X)$ has codimension $\geq g(\tilde{C})$.*

Theorem 2.20. [18, Theorem 1.3] *Let $\pi : X \to Y$ be a primitive type III contraction of a non-singular Calabi-Yau threefold X contracting a divisor E to a smooth curve C. If $g(C) \geq 1$, then Y is smoothable.*

We do not quote the proof of the theorems. Instead we discuss certain aspects of the deformation of the contraction to provide a better understanding of our situation. Let $\mathscr{X} \to \Delta$ be a Kuranishi family for $X = X_0$. By [18, Proposition 1.2] and [32, Proposition 6.5] we know that the locus in Δ where E deforms in the family of $\mathscr{X}$ is of codimension g. When X contains a smooth surface ruled over a curve of genus $g > 1$ then for every fibre Z of E and some disc around 0 we have that every X_t in the family contains a curve which is a deformation of Z. More specifically, following discussion in [42, Section 4], fibers over $2g - 2$ points of C will extend sideways in the family of deformations $\mathscr{X}$. From this we conclude that whenever $g > 1$, the primitive type III contraction of $\pi : X_0 \to Y_0$ deforms to a family of type I contractions $\pi_t : X_t \to Y_t$ of $2g - 2$ fibers. This means that for general t the contraction $X_t \to Y_t$ is a small (type I) primitive contraction and unless $\mathrm{Sing}(Y_t)$ consists of exactly one ordinary double point, Y_t is smoothable. As E is normal Y_t cannot have only one ODP so the general Y_t is smoothable. In the case when $g(C) = 1$, we have that neither E nor L deforms in the family $\mathscr{X}$ and that $X_t \to Y_t$ is an isomorphism for general t and so Y is smoothable.

We recall the [39, Conjecture 6.7] that a sufficiently general transition arising from the type III contraction deforms to a conifold transition. From the above discussion it follows that this conjecture holds for surfaces ruled over smooth curves of genus > 1. Namely we have proved:

Corollary 2.21. *Let $T(X, Y, \tilde{Y})$ be a geometric transition. Assume that the contraction $\pi : X \to Y$ contracts a smooth surface E ruled over a smooth curved C of genus > 1. Then T deforms to a conifold transition.*

3. Calabi-Yau threefolds containing a cone and a triple point

The main goal of this section is to construct Calabi-Yau threefolds X admitting type three contractions. Namely we construct Calabi-Yau threefolds containing a cone E over some curve, such that the vertex of the cone is an ordinary triple point of $\bar{X}$ which we denote O. A general threefold in question will also have ordinary double points lying on this cone. We perform resolution of singularities of $\bar{X}$, first by blowing up O and then by performing a small blow-up of double points. Both of those do not affect the canonical class of X and thus we obtain another Calabi-Yau threefold $\tilde{X}$ contating a smooth ruled surface $\tilde{E}$ arising as the strict transform of the cone E. Then we find a contraction morphism $\pi : \tilde{X} \to Y$ that takes $\tilde{E}$ to a curve. Depending on the genus of the curve the variety Y may be smoothable and in case it is we obtain another example of a smooth Calabi-Yau threefold that we denote $\tilde{Y}$.

Throughout the thesis we label the variables in $\mathbb{P}^4$ or $\mathbb{P}(1:1:1:1:2)$ ($\mathbb{P}^5$) with x, y, z, t, u (w). When we discuss the degrees of the curves we mean the degree in $\mathbb{P}^3$ unless otherwise stated.

3.1. **Quintic threefolds containing a ruled surface and a triple point.** We define $\bar{X}_5 = V(F)$ where

$$F = u^2F_3 + uF_4 + F_5$$

is a homogenous polynomial of degree 5 with $V(F_i)$'s being smooth degree i surfaces in $\mathbb{P}^3$ (and so F_i are independent of u) containing a smooth curve C. As our construction requires this curve to be simultaneously contained in degree $3, 4$ and 5 surfaces, we limit ourselves to the curves of degree ≤ 15. Choosing a higher degree curve would mean that $V(F_3) \subset V(F_4, F_5)$ and so in particular

$$V(F) = V(F_3) \cup V(u^2 + uF_1 + F_2)$$

with $F_4 = F_1F_3$ and $F_5 = F_2F_5$ would be reducible. In choosing the particular curves, we follow the classification of possible genus-degree combinations of curves in $\mathbb{P}^3$ in [20]. We do not claim this list is complete. We construct the curves as ones living on a cubic surface in $\mathbb{P}^3$ either by intersecting it with some other surface or by pulling back a curve on $\mathbb{P}^2$ under the blow up of 6 points in general position as for example in [22, V 4.7]. In Table 2 we provide the list of curves that serve as a basis for our construction. The table gives the curve coordinates in Picard group of a cubic surface as well as its genus $g(C)$ and canonical class K_C. In general by C_i we mean a curve of degree i. The curve $C_{a,b}$ is obtained as an intersection of degree a, b surfaces in $\mathbb{P}^3$ and TC is a twisted cubic. Whenever we discuss more than one curve of a given degree we add a capital letter to the index to distinguish them. When we refer to the specific quintic $\bar{X}_5$ (or its resolution X_5) containing a given curve C_i we denote it by $\bar{X}_{5(i)}$ (resp $X_{5(i)}$). The same applies for $Y_{5(i)}$ and $\tilde{Y}_{5(i)}$ - threefolds obtained by a contraction and respective smoothing. On the other hand when the context is clear we drop 5 from the index for brevity.

We denote by E the cone over $C \subset \bar{X}$ and by O its vertex $[0:0:0:0:0:1]$ being a triple point of $\bar{X}$. We assume that there are no additional one-dimensional components of $V(F_3) \cap V(F_4) \cap V(F_5) \subset \mathbb{P}^3$ apart from C. It may happen, though, that there are some isolated points in the intersection $V(F_3) \cap V(F_4) \cap V(F_5)$ which we call excess points. They give rise to lines passing through the triple point of $\bar{X}$ that are otherwise disjoint from $\bar{E}$. Furthermore, the threefold $\bar{X}$ can have other singular points than the vertex of the cone but we show that they are ordinary double points admitting a small resolution. To that aim let Λ be a complete linear system of quintic threefolds containing a cone $\bar{E}$ over a given curve C with an ordinary triple point at the vertex of the cone.

Theorem 3.1. *Let C, a smooth irreducible curve, be the only one-dimensional component of $V(F_3) \cap V(F_4) \cap V(F_5) \subset \mathbb{P}^3$. Let $\bar{E}$ be the cone over C in $\mathbb{P}^4$. Let Λ be the complete linear system of quintic threefolds containing $\bar{E}$ with an ordinary triple point at the vertex of $\bar{E}$. A general element $\bar{X}$ of Λ outside of the vertex of the cone has at worst nodes as singularities. Furthermore, the singular points of $\bar{X}$ lie on the cone $\bar{E}$.*

Proof. Let $\pi : \tilde{\mathbb{P}^4} \to \mathbb{P}^4$ be the blow-up of the $\mathbb{P}^4$ along $\bar{E}$ and let E be the exceptional divisor. We will show that $\bar{E}$ is the base locus of Λ - the complete system of quintic threefolds containing $\bar{E}$ and thus obtain that $|\pi^*\Lambda - E|$ is base-point free outside of the $Z := \pi^{-1}(O)$.

From the assumption $V(F_3) \cap V(F_4) \cap V(F_5) \subset \mathbb{P}^3$ has no one-dimensional components outside of C. It is straightforward that

$$X = V(F) = V(x^2F_3 + xF_4 + F_5) \in \Lambda$$

Consider

$$X_m = V(F_m)$$

with

$$F_m = (u^2F_3 + uF_4 + F_5 + F_3G_m)$$

where G_m is a degree two polynomial in x, y, z, t. We see $X_m \in \Lambda$. The restriction of $V(G_m)$ to the cubic surface $V(F_3) \subset \mathbb{P}^3$ can be considered an element of an ample linear system $|2H|$ with empty base locus and thus there are no isolated excess lines in the base locus of Λ meaning $\bar{E} = \mathrm{Bs}\Lambda$ as required. Note that $Z = \pi^{-1}(O)$ is in the base locus of the system $|\pi^*\Lambda - E|$.

Consider the morphism $\Phi : \tilde{\mathbb{P}^4} \to \mathbb{P}^N$ given by the linear system $\pi^*\Lambda - E$. Restrict this morphism to $\Phi|_{\tilde{\mathbb{P}^4}\setminus Z} : \tilde{\mathbb{P}^4}\setminus Z \to \mathbb{P}^N$. For each point p of $\bar{E}$ outside of O we obtain that this restriction embeds $\pi^{-1}(p)$ as a line $\mathbb{P}^1$ in $\mathbb{P}^N$. Let $\mathbb{P}^{N^*}$ be the dual projective space to $\mathbb{P}^N$. We work in the product $\bar{E}\setminus O \times \mathbb{P}^{N^*}$ with natural projections π_1,π_2. We define $I = \{(p, H) : \pi^{-1}(p) \subset H\}$. It follows that $\pi_1^{-1}(p)$ is the set of hyperplanes containing a fixed line so the dimension of $\pi_1^{-1}(p)$ is $N - 2$ which says that the set of quintics having point p as a singularity is of codimension 2. Furthermore we get

$$\dim I = \dim \bar{E}\setminus O + N - 2 = 2 + N - 2 = N$$

and thus $\dim \pi_2^{-1}(H) = 0$ for general H. We have that $\pi_1(\pi_2^{-1}(H))$ is the singular locus of the quintic $\bar{X}$ whose small resolution is birational to the hyperplane section $H \cap \Phi(\tilde{\mathbb{P}}^4)$, and so for a general H the singular locus is at most zero-dimensional.

At every point of the cone $\bar{E}$ which is not its vertex we can find quintics which have different tangent directions. To see that, observe that if $\bar{X} = V(F)$ is the quintic containing $\bar{E}$ and P is the point on $\bar{E}$ we have $F(P) = 0$ where

$$F = u^2 F_3 + uF_4 + F_5$$

and F_i are homogeneous polynomials of degree i as above. We can substitute $\tilde{F}_4 = F_4 + F_1 F_3$ and $\tilde{F}_5 = F_5 + F_2 F_3$ where F_1 and F_2 are general linear and quadratic homogeneous polynomials thus obtaining equation of another quintic containing the cone $\bar{E}$ but having different tangent direction at P. For any $\alpha \in \{x, y, z, t\}$ we obtain

$$\frac{\partial \tilde{F}}{\partial \alpha} = \frac{\partial F}{\partial \alpha} + F_3 (u \frac{\partial F_1}{\partial \alpha} + \frac{\partial F_2}{\partial \alpha}) + \frac{\partial F_3}{\partial \alpha}(uF_1 + F_2).$$

Since F_3 is the equation of a smooth cubic surface it is enough that we have $F_2(P) \neq -uF_1(P)$ for the statement to be true. This construction also shows that for each point $P \in \bar{E}$ there exists a quintic containing $\bar{E}$ which is not singular at P. By the Bertini theorem, we conclude that general $\bar{X}$ has at worst double points as singularities but for the vertex of the cone. What is more the singularities lie on the cone $\bar{E}$ and outside of $\bar{E}$ the general $\bar{X}$ is smooth.

By following the argument of [28, Theorem 4.4], we conclude that a generic element $\bar{X}$ of Λ has only singularities of cA type as for every point P of $\bar{E}$ but for the vertex we can find $\bar{X}_P$ which is not singular at P. From [9, Claim 2.2], we have that $P \in \bar{E}$ is a singular point of a general $\bar{X} \in \Lambda$ if and only if $\bar{X}'$ such that $\pi(\bar{X}') = \bar{X}$ contains the whole fiber $\pi^{-1}(P)$.

Now let P be a singular point (different than the vertex of the cone) of a general quintic $\bar{X}$ containing $\bar{E}$. We blow up the whole of $\bar{E}$ and obtain $\mathbb{P}^1$ as a fiber of π over P. From this and the previous paragraph, we see that a general element X of Λ admits a small resolution (outside O) with $\mathbb{P}^1$ as the exceptional locus. Using Bertini we see that a general element of $|\pi^*\Lambda - E|$ cuts E along a nonsingular surface and so the normal bundle of C contains a subbundle $\mathscr{O}_C(-1)$. Following the argument from [24, Theorem 2.1] we know that in a small resolution of a cA type singularity there is a curve with normal bundle

$\mathscr{O}(-1)\bigoplus\mathscr{O}(-1)$ or $\mathscr{O}\bigoplus\mathscr{O}(2)$. In our case this has to be the former of the two which means the singularities of a generic $\bar{X}$ outside the vertex of the cone are ordinary double points. □

Fact 3.2. *For nearly all quintic threefolds $\bar{X}$, we have found that the intersection of $V(F_3), V(F_4)$ and $V(F_5)$ consists of the curve C and a finite number of points (we call them excess points). Thus, these quintic threefolds contain not only a cone $\bar{E}$ but also a set of lines all passing through the vertex O. The exceptions are quintics containing curves $C_{10B}, C_{11}, C_{3,4}$ and $C_{3,5}$ as for them there are no excess points and thus no excess lines. Denote the hyperplane section of cubic surface by H_D and the class of a curve on D by C. Then the number of excess points can be calculated using the excess intersection formula:*

$$(4H_D - C).(5H_D - C) = 20H_D^2 - 9H_D.C + C^2 = 60 - 9deg(C) + C^2$$

which is indeed 0 in the cases listed above.

3.2. **Complete intersection threefolds containing a cone and a triple point.** We discuss two complete intersection threefolds namely $\bar{X}_{2,4}$ and $\bar{X}_{3,3}$ in $\mathbb{P}^5$. Again we want to focus on those containing a cone over a smoth curve with a triple point at the vertex of the cone. We being with describing the construcion. We adapt similar notation convention as in the case of quintic threefolds.

3.2.1. *Complete intersection $X_{2,4}$.* We begin with a smooth curve $C \subset \mathbb{P}^4$. We find smooth varieties A_i's of degrees $1, 2, 3$ and 4 in $\mathbb{P}^4$ containing this curve. Let F_i's be polynomials in 5 variables defining A_i's. Define $X_2 = V(wF_1 + F_2) \subset \mathbb{P}^5$ and $X_4 = V(wF_3 + F_4) \subset \mathbb{P}^5$. Then $O = [0 : 0 : 0 : 0 : 0 : 1]$ is the triple point of X_4 and a smooth point on X_2 and as such it is an OTP for their complete intersection $\bar{X}_{2,4}$. The construction shows that $\bar{X}_{2,4}$ contains a cone E over a curve C whose vertex is the discussed OTP. The construction calls for the curve to be actually contained in $\mathbb{P}^3 \cong V(F_1)$. Since the complete intersection of $V(F_1)$, $V(F_2)$ and $V(F_4)$ is a degree 8 curve we restrict ourselves to curves with degree ≤ 8. Moreover, as this curve needs to be contained in $V(F_3)$ this limits the family of curves even further. If we write $F_3 = F_1G_2 + F_2G_1$ with G_i homogeneous of degree i then the point O is not an ordinary triple point on $X_{2,4}$ as the exceptional divisor D over O contains a singular curve $V(F_2) \cap V(G_1)$. Thus the highest degree curve that we can use in our construction is the complete intersection

$$C_6 = V(F_1) \cap V(F_2) \cap V(F_3).$$

In Table 5 we provide the list of curves that serve as a basis for our construction.

Theorem 3.3. *Let C, a smooth irreducible curve be the only one-dimensional component of the intersection of smooth hypersurfaces*

$$V(F_1) \cap V(F_2) \cap V(F_3) \cap V(F_4) \subset \mathbb{P}^4.$$

Let $\bar{E}$ be the cone over C in $\mathbb{P}^5$. Assume that a general quadric fourfold containing $\bar{E}$ are smooth outside the vertex of $\bar{E}$. Then a general complete intersection threefold $\bar{X}_{2,4}$ containing $\bar{E}$ with a triple point at the vertex of the cone has at worst nodes as the remaining singularities. Furthermore, the singular points of $\bar{X}_{2,4}$ lie on the cone $\bar{E}$.

Proof. The proof mimics the proof of 3.1. We describe the steps that need slight modification.

Let $X := X_2$ be a chosen, nonsingular quadric fourfould containing $\bar{E}$. Let $\pi : \tilde{X}_2 \to X_2$ be the blow-up of the X_2 along $\bar{E}$ and let E be the exceptional divisor. We will show that $\bar{E}$ is the base locus of Λ - the complete system of complete intersection threefolds of multidegree $(2,4)$ containing $\bar{E}$ and thus obtain that $\pi^*\Lambda - E$ is base-point free outside of the $Z := \pi^{-1}(O)$.

From the assumption

$$V(F_1) \cap V(F_2) \cap V(F_3) \cap V(F_4) \subset \mathbb{P}^4$$

has no one-dimensional components outside of C. It is straightforward that $\bar{X}_{2,4} \in \Lambda$. Consider $\bar{X}_{4b} = (wF_3 + F_4 + F_2G_b)$ where G_b is a degree two polynomial in x, y, z, t, u. We see $\bar{X}_{2,4b} \in \Lambda$. The restriction of $V(G_b)$ to the cubic surface $V(F_3) \subset \mathbb{P}^3$ can be considered an element of an ample linear system $|2H|$ with empty base locus and thus there are no isolated excess lines in the base locus of Λ meaning $\bar{E} = \mathrm{Bs}\Lambda$ as required. Note that $Z = \pi^{-1}(O)$ is in the base locus of the system $\pi^*\Lambda - E$.

Consider the morphism $\Phi : \tilde{X} \to \mathbb{P}^N$ given by the linear system $\pi^*\Lambda - E$. Restrict this morphism to $\Phi|_{\tilde{X}\setminus Z} : \tilde{X}\setminus Z \to \mathbb{P}^N$. For each point p of $\bar{E}$ outside of O we obtain that this restriction embeds $\pi^{-1}(p)$ as a line $\mathbb{P}^1$ in $\mathbb{P}^N$. Let $\mathbb{P}^{N^*}$ be the dual projective space to $\mathbb{P}^N$. We work in the product $\bar{E}\setminus O \times \mathbb{P}^{N^*}$ with natural projections π_1,π_2. We define $I = \{(p, H) : p \subset H\}$. It follows that $\pi_1^{-1}(p)$ restricted to I is the set of hyperplanes containing a fixed line so the dimension of $\pi_1^{-1}(p)$ is $N - 2$ which says that the set of $X_{2,4}$ singular at p is of codimension 2. Furthermore we get

$$\dim I = \dim \bar{E}\setminus O + N - 2 = 2 + N - 2 = N$$

and thus $\dim \pi_2^{-1}(H) = 0$ for general H. We have that $\pi_1(\pi_2^{-1}(H))$ is the locus where $\bar{X}_{2,4}$ is singular and so for a general H this locus is at most zero-dimensional.

At every point of the cone $\bar{E}$ which is not its vertex we can find quartic fourfolds with different tangent directions. Using similar equations to the ones above observe that if $\bar{X}_4 = V(wF_3 + F_4)$ is the quartic containing $\bar{E}$ we can write

$$\bar{X}_{4ab} = (wF_3 + F_4 + F_1H_a + F_2G_b)$$

with H_a general linear and G_b general quadric equation independent of w. Note that $\bar{X}_{4ab}$ contains the cone $\bar{E}$ but has different tangent direction at P - a point lying on $\bar{E}$. This again shows that for each point $P \in \bar{E}$ there exists an $\bar{X}_{2,4}$ containing $\bar{E}$ which is not singular at P. By the Bertini theorem, we conclude that general $\bar{X}_{2,4}$ has at worst double points as singularities but for the vertex of the cone. What is more the singularities lie on the cone $\bar{E}$ and outside of $\bar{E}$ the general $\bar{X}_{2,4}$ is smooth.

The rest of the proof is identical to the last two paragraphs of the proof of Theorem 3.1. □

We assumed that general X_2 and X_4 containing $\bar{E}$ are smooth outside of the vertex of the cone. We can see that this is not always the case. For instance let C_3 be a complete intersection of two hyperplanes H_1, H_2 and a cubic threefold in $\mathbb{P}^4$. We can assume $H_1 = V(x), H_2 = V(y)$. Then any quadric fourfold containing a cone over C_3 in $\mathbb{P}^5$

has equation of the form $F_2 = xA_1 + yB_1$ with A_1, B_1 linear polynomials and so is singular along a line $x = y = A_1 = B_1 = 0$. For all the curves in Table 5 we have managed, using Macaulay2, to find smooth X_2 and smooth X_4 containing the cone $\bar{E}$ over it in $\mathbb{P}^5$. This shows that a general X_2 and X_4 containing $\bar{E}$ are smooth and so the theorem works in cases we analyse.

Fact 3.4. *Similarily as in the case of quitinc threefolds and using notation from Fact 3.2 we obtain excess lines contained in $X_{2,4}$. Straightforward calculations show there are*

$$(2H_D - C).(4H_D - C) = 40 - 6deg(C) + C^2$$

of them.

3.2.2. *Complete intersection $X_{3,3}$.* We begin again with a smooth curve $C \subset \mathbb{P}^4$. We find four hypersurfaces $V(A_3)$ and $V(B_1), V(B_2), V(B_3)$ in $\mathbb{P}^4$ containing this curve. Now, let $X_3 = V(A_3) \subset \mathbb{P}^4$ and $Y_3 = V(x_5^2 B_1 + x_5 B_2 + B_3) \subset \mathbb{P}^5$. We see that X_3 has a triple point at $O = [0:0:0:0:0:1]$ and Y_3 is smooth there thus giving us an ordinary triple point on $\bar{X}_{3,3} = X_3 \cap Y_3$ with a cone E over a curve C. In this case our curve C can be of at most degree 9 with $C_9 = V(F_1) \cap V(B_3) \cap V(A_3)$ with B_2 being a zero polynomial. Thus in our construction we limit ourselves to curves of deg ≤ 9. In Table 6 we provide the list of curves that serve as a basis for our construction.

In all cases we analyse one of the cubic fourfolds is a cone itself with a triple point being its vertex (and smooth outside of it) while the other is smooth. We can denote them X_3 and Y_3 respectively. We want to obtain a result similar to 3.1 and 3.3.

Theorem 3.5. *Let C, a smooth irreducible curve be the only one-dimensional component of the intersection of smooth hypersurfaces*

$$V(F_1) \cap V(F_2) \cap V(F_3) \cap V(G_3) \subset \mathbb{P}^4.$$

Let $\bar{E}$ be the cone over C in $\mathbb{P}^5$. Let $X_3 = V(G_3)$ be a cone in $\mathbb{P}^5$ and let a general cubic fourfold Y_3 containing $\bar{E}$ be smooth outside the vertex of $\bar{E}$. Then a complete intersection threefold $\bar{X}_{3,3}$ of X_3 and Y_3 containing $\bar{E}$ with a triple point at the vertex of the cone has at *worst nodes as the remaining singularities. Furthermore, the singular points of $\bar{X}_{3,3}$ lie on the cone $\bar{E}$.*

Proof. The proof is similar to the one of 3.3 with minor adjustments. □

Fact 3.6. *Again using notation from Fact 3.2 the number of excess lines in $\bar{X}_{3,3}$ is*

$$(2H_D - C).(3H_D - C) = 30 - 5deg(C) + C^2.$$

3.3. **Sextics in $\mathbb{P}(1:1:1:1:2)$ containing a ruled surface and a triple point.** We move on to the last case we consider, namely sextic threefold X_6 in the weighted projective space $\mathbb{P}(1:1:1:1:2)$. We use x, y, z, t to denote weight 1 an u to denote the weight 2 variable. Sometimes we write $\mathbb{P}$ to denote the WPS if the context is clear. We use as a main reference regarding the weighted projective spaces.

Before we delve into details we establish some facts about the notation and the general properties of weighted projective space $\mathbb{P}(1:1:1:1:2)$. Note that in this space the divisors defined by degree one equations, for example $V(x)$ are not Cartier. For every degree one equation F_1 we see that $V(F_1)$ necessarily passes through the singular point $[0:0:0:0:1]$ of the WPS. On the other hand the general element of $|\mathscr{O}_{\mathbb{P}}(2)|$, that is divisor equivalent to $u = 0$, will avoid the singular point of WPS and be Cartier. We keep the notation $2H$ for elements of this system to be consistent with the weighted degree of their defining equations. We keep in mind that $V(F_2)$ for a general degree 2 polynomial in $\mathbb{P}(1:1:1:1:2)$ is isomorphic to $\mathbb{P}^3$. In particular this applies to $V(t)$. Also note that even though H is not Cartier on WPS it is Cartier when limited to X_6 as H_X avoids the singular point of the projective space.

The system $|\mathscr{O}(2)|$ provides a natural embedding of WPS into $\mathbb{P}^{10}$. The image of this embedding is the cone over a second Veronese embedding $v(\mathbb{P}^3) \hookrightarrow \mathbb{P}^9$. In this setting the sextic hypersurface can be considered as a triple section of a this cone and so a triple cover of $v(\mathbb{P}^3)$.

We consider $\bar{X}_6$ with the defining equation

$$F = t^3 A_3 + t^2(uB_2 + B_4) + t(u^2 C_1 + uC_3 + C_5) + u^3 + u^2 D_2 + uD_4 + D_6$$

with A_i, B_i, C_i, D_i weighted homogeneous polynomials of degree i independent of t, u. For brevity let us denote the polynomials by A, B, C, D so that

$$F = t^3 A + t^2 B + tC + D.$$

Consider a smooth curve C contained in the intersection

$$V(A) \cap V(B) \cap V(C) \cap V(D) \in \mathbb{P}(1:1:1:2).$$

The ruled surface $\bar{E}$ over C defined as in previous examples is contained in $\bar{X}_6 \in \mathbb{P}(1:1:1:1:2)$.

Note, that as we want $\bar{X}_6$ and the curve C to avoid the singular point of WPS we require u^3 to appear in F_6. Also, since at $t = 1$ we want locally the equation to be of degree 3 we cannot have terms with u^2 in B or u in A. Setting $t = 1$ we see that the equation of a tangent cone at O is given by

$$G_3 = A_3 + uB_2 + u^2C_1 + u^3.$$

We are more limited with the choice of curves as they need to avoid the singular point of $\mathbb{P}(1:1:1:2)$ and so a term u^m has to appear in at least one equation generating them for some $m > 0$. At this point we reach another obstruction. As equations defining C have to be weighted homogeneous, then whenever we have u^m in the equation we see that it is of degree $2m$ and so the other terms appearing in this polynomial are of non-weighted degrees between $m+1$ and $2m$. We can write $F_{2m} = u^m + G_{2m}(x, y, z, u)$. The surface $\bar{E}$ in $\mathbb{P}(1:1:1:1:2)$ is defined by the same equations as C. When we work in the affine chart $t = 1$ variables are no longer considered with weights and so one of the equations defining the tangent cone to $\bar{E}$ at O is just u^m (as other terms in F_{2m} are of higher, non-weighted degree). This means that after the blowup of O the intersection of E and D would be a non-reduced curve and so E would not be a surface ruled over a smooth curve as required. We are thus left with C being defined by the equation of weighted degree 2 in which the term u appears in the first power. In particular we can assume that the curves in question are contained in $V(u)$ and so - by analogy with $V(u)$ being isomorphic to $\mathbb{P}^3 \subset \mathbb{P}(1:1:1:1:2)$ - can be treated as living in $\mathbb{P}^2$. As this curve has to be contained in a cubic surface this leaves us with three possible curves, that is degree 1, 2 and 3 smooth planar curves. In this setting the analogue of Theorem 3.1 works and so we obtain:

Theorem 3.7. *Let C, a smooth irreducible curve as above, be the only one-dimensional component of*

$$V(A) \cap V(B) \cap V(C) \cap V(D) \subset \mathbb{P}(1:1:1:2)$$

such that $C \subset V(u)$. Let $\bar{E}$ be the surface defined by the same equations in $\mathbb{P}(1:1:1:1:2)$ as C in $\mathbb{P}(1:1:1:2)$. Let Λ be the complete linear system of sextic threefolds containing $\bar{E}$ with an ordinary triple point at $O = [0:0:0:0:1:0]$. A general element $\bar{X}$ of Λ outside of the triple point has at worst nodes as singularities. Furthermore, the singular points of $\bar{X}$ lie on the surface $\bar{E}$.

Note that contrary to the previous sintuations in the case of a sextic we do not expect, and in our examples, do not obtain any excess components of the intersection of surfaces in $\mathbb{P}(1:1:1:2)$. The intersection of 4 surfaces in $\mathbb{P}(1:1:1:2)$ is expected to be empty and so in it there are no additional points outside of C.

4. Intersection theory on Calabi-Yau threefolds containing a ruled surface

We want to discuss the intersection theory on Calabi-Yau threefolds that we contract. Let $\bar{X}$ be a Calabi-Yau threefold containing a cone $\bar{E}$ over a curve C with triple point O at its vertex as above. We consider threefold X obtained after, first, blowing up the point O and then performing a small blow-up of nodes as ilustrated in the following diagram.

$$\begin{array}{ccc} E & \hookrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ \tilde{E} & \hookrightarrow & \tilde{X} \\ \downarrow & & \downarrow{\scriptstyle \bar{\pi}} \\ \bar{E} & \hookrightarrow & \bar{X} \end{array}$$

4.1. **Threefolds containing a ruled surface from quintic threefolds.** Let $\pi : X \to \bar{X}$ be the composition of two blow-ups. Let H_X be the strict transform of a hyperplane section of $\bar{X} \subset \mathbb{P}^4$ and H'_X the strict transform of the hyperplane section of $\bar{X}$ passing through O. Let E be the strict transform of the cone $\bar{E}$ over the curve C on $\bar{X}$ and let $D = \pi^{-1}(O)$ be the exceptional divisor, that is a smooth cubic surface contained in X coming from the blow-up of the triple point O on $\bar{X}$ We denote by l the fibre of E. We write g for genus of C. In this part we want to show the following theorem.

Theorem 4.1. *Let X be as above and assume additionally that $h^{1,1}(X) = 3$. Then $Pic_{\mathbb{Q}}(X)$ is generated by three divisors: H_X, D, and E.*

As $h^{1,1}(X)$ in the case of irreducible Calabi-Yau threefold is the rank of its Picard group this theorem reduces to saying that the divisors in question are linearly independent. To prove this we will need to calculate the intersections of divisors with themselves and with curves on X. We provide these in the following lemmas.

Lemma 4.2. *On X, a Calabi-Yau threefold with notations as above, we have:*

(1) $H_X.D = 0$;

(2) $H_X.E = C + deg(C)l$;

(3) $D.E = C$;

(4) $E^2 = K_E = -2C + (K_C - deg(C))l$;

(5) $E^3 = K_E^2 = 8(1-g)$;

(6) $D^2 = K_D$;

(7) $D^3 = 3$.

Proof. General hyperplane section of $\bar{X}$ misses point O and thus H_X misses D by properties of pullback which proves (1). The exceptional divisor D intersected with E is precisely the base curve C [22, V. Proposition 2.11.4] and thus $D.E = C$ hence (2). Hyperplane passing through O cuts $\bar{E}$ in exactly $deg(\bar{E}) = deg(C)$ fibers. By [12, Corollary 9.12], we can think of a class D as a class $H_X - H'_X$ and thus $H_X \sim D + H'_X$ and so

$$H_X.E = (D + H'_X).E = C + deg(C)l.$$

We calculate E^2 from the adjunction formula.

$$K_E = (K_X + E)|_E = E^2$$

and similarily

$$K_E^2 = (K_X + E)^2|_E = (K_X^2 + 2K_XE + E^2)|_E = E^2|_E = E^3$$

since $K_X = 0$. Again by [22, V. Corollary 2.11] we know that $E^3 = 8(1-g)$. Similarily from adjunction $D^2 = K_D$ and since D is a smooth cubic surface $K_D \sim -H_D$ where H_D is a hyperplane section of a cubic. This is a standard use of adjunction as

$$K_D \sim \mathscr{O}_D(-3 - 1 + deg(D)) = \mathscr{O}_D(-1) \sim -H_D.$$

Since $D^3 \sim K_D^2$ we obtain $D^3 \sim (-H_D)^2 = 3$ which completes the proof. □

We use $l_{|E}$ to denote the class of a fiber or a ruled surface E in the $N_1(E)_{\mathbb{R}}$. We use analogous notation for the other curves on E and D.

Lemma 4.3. *We have*

(1) $l_{|E}^2 = 0$;

(2) $C_{|E}^2 = -deg(C)$;

(3) $l_{|E}.C_{|E} = 1$;

(4) $l_{|E}.E = -2$;

(5) $C_{|E}.E = K_C + deg(C)$;

(6) $D.C_{|D} = -deg(C)$.

Proof. Points (1)-(4) are standard facts for ruled surfaces and for example found in [22, V. Section 2]. To obtain point (5) we can restrict $C_{|E}.E_{|E}$ and thus

$$C_{|E}.K_E = C.(-2C + (K_C - deg(C)l) = 2deg(C) + K_C - deg(C) = K_C + deg(C).$$

By similar argument

$$D.C_{|D} = K_D.C_{|D} = -H_D.C_{|D} = -deg(C).$$

□

We can proceed with

Proof of Theorem 4.1. We need to show that H_X, D and E are linearly independent in $Pic_{\mathbb{Q}}(X)$. To this end assume that

$$T = \alpha H_X + \beta D + \gamma E$$

has zero intersection with classes of curves

$$\{H_X^2, H_X.E, D^2, D.E, E^2\}$$

for $\alpha, \beta, \gamma \in \mathbb{Q}$. We obtain the following set of equalities

(1) $5\alpha + d\gamma = 0$

(2) $d\alpha + (deg(K_C) - d)\gamma = 0$

(3) $3\beta - d\gamma = 0$

(4) $-d\beta + (deg(K_C) + d)\gamma = 0$

(5) $(deg(K_C) - d)\alpha + (deg(K_C) + d)\beta + (8 - 8g)\gamma = 0$.

There are 3 possible solutions. Either $\alpha = \beta = \gamma = 0$ which means H_X, D and E are linearly independent as desired, $d = 0$ which is a contradiction or $d = 15$ and $g(C) = 31$ which means the curve C is the complete intersection of a cubic and a quintic surface meaning $E = 3H_X - 5D$ and so the Picard group of X is no longer of rank 3. □

Let r denote the class of $\mathbb{P}^1$ coming from the small blow-up of a node of X lying on the ruled surface. Recall that $V(F_3) \cap V(F_4) \cap V(F_5)$ contains C and some excess points, thus $\bar{X}$ contains a cone $\bar{E}$ over C and lines over excess points. We use t to denote the class of the strict transform of one of these lines in X.

Lemma 4.4. *On X as above we have*

(1) $H_X.r = 0$
(2) $D.r = 0$
(3) $E.r = 1$

Furthermore, we have the numerical equivalence of curves

(1) $t \sim l + 2r$.

Proof. As r is the $\mathbb{P}^1$ replacing the point lying directly on $\bar{E}$ where $\bar{E}$ is smooth the equality $E.r = 1$ is clear. General hyperplane section of $\bar{X}$ misses this point giving $H_X.r = 0$ and since D is the blow-up of the vertex of $\bar{E}$ it misses r as well. One can easily derive the numerical equivalence class of t by looking at the Table 1 which describes the intersection of curves and divisors on X. □

We need to consider the curves on X that lie on the cubic surface D. Since $H_{X|D} = 0$, $D_{|D} = -H_D$, $E_{|D} = C_0$ we see that restriction of $Pic_{\mathbb{Q}}(X)$ to $Pic_{\mathbb{Q}}(D)$ is at most two dimensional (it may happen that the curve C_0 is the multiple of a hyperplane section of D and then it is one dimensional). We describe the intersection of curves on D with divisors of X. We have $H_D = 3h - \Sigma_{i=1}^{6} e_i$ and $C_0 = ah - \Sigma_{i=1}^{6} b_i e_i$ for $a, b_i \in \mathbb{Z}$. Then we show

Lemma 4.5. *For h, and e_i, $i \in 1, \ldots, 6$ generators of $Pic(D)$ we have the following equivalences on X:*

(1) $h \sim \frac{-3}{-3a+\Sigma b_i} C_0 + (a + \frac{3(a^2-\Sigma b_i^2)}{-3a+\Sigma b_i})r$
(2) $e_i \sim \frac{1}{3a-\Sigma b_i} C_0 + (b_i - \frac{a^2-\Sigma b_i^2}{3a-\Sigma b_i})r.$

Proof. We use Table 1 again to deduce the equivalences. The equivalence classes of h and e_i are well-defined as long as $3a - \Sigma b_i \neq 0$, but this expression is exactly $deg(C)$ and so never 0. □

4.2. **Threefolds containing a ruled surface from complete intersection threefolds.** The discussion of intersection theory for $X_{2,4}$ and for $X_{3,3}$ is analogous to that for X_5. The only difference occurs in the proof of 4.1. In case of $X_{2,4}$ intersection of

$$T = \alpha H_X + \beta D + \gamma E$$

with $\{H_X^2, H_X.E, D^2, D.E, E^2\}$ the result of first intersection is different while the remaining four are the same as in the proof 4.1.

(1) $8\alpha + d\gamma = 0.$

Now there are only 2 possible solutions. Either $\alpha = \beta = \gamma = 0$ which means H_X, D and E are linearly independent as desired, $d = 0$ which is a contradiction. For $X_{3,3}$ we get

(1) $9\alpha + d\gamma = 0.$

The system of linear equations again has 2 solutions. Apart from the trivial one we have $d = 9$, $g(C) = 10$ which happens when C is the complete intersection of two cubics and so E is equivalent to the hyperplane section of $X_{3,3}$.

4.3. **Threefolds containing a ruled surface from sextic threefolds.** The intersection theory for X_6 obtained by resolving singularities of a sextic in $\mathbb{P}(1:1:1:1:2)$ is completely analogous to the one discussed above. The proof of the Theorem 4.1 applied here we solve

(1) $6\alpha + d\gamma = 0.$

This system of equations again has no nontrivial solutions and so we see that H_X, D and E are linearly independent on X_6 as desired.

4.4. **Kähler cone of a Calabi-Yau threefold containing a ruled surface and a cubic surface with Picard rank 3.** For simplicity we write H instead of H_X to denote the pullback of the hyperplane section of $\bar{X}$ on X. We claim the following:

Theorem 4.6. *Let X be X_5, $X_{2,4}$, $X_{3,3}$ or X_6 as above with Picard group generated by H, D and E. Then the closure $\bar{K}$ of the Kähler cone K is contained in the convex hull of three rays. Two of the rays are generated by divisors H and $H-D$ and don't lie on the cubic cone W. Moreover, these two rays belong to $\bar{K}$ and span one of its faces.*

Proof. Clearly H is nef on X as it is a pullback of a hyperplane section of $\bar{X}$. It has zero intersection with C_0, r and every curve contained in D and positive intersection with other curves. We can think of $H-D$ as a pullback of a hyperplane section passing through the point O. Thus it is enough to note that $H-D$ has a non-negative intersection with curves on D as

$$(H-D)_{|D} = H_D$$

and with curves on E as

$$(H-D)_{|E} = (C_0 + deg(C).l) - C_0 = deg(C).l.$$

It also has a zero interesection with r. Observe that for X_5 ($X_{2,4}$, $X_{3,3}$, X_6) we have $H^3 = 5 > 0$ ($8, 9, 6 > 0$ for respectively) and

$$(H-D)^3 = H^3 + 3H^2.D + 3H.D^2 + D^3 = 5 - 3 = 2 > 0$$

($5, 6, 3 > 0$ respectively) so indeed these divisors do not lie on the cubic cone W. If we write divisors in $Pic(X)$ as $\alpha H + \beta D + \gamma E$, we see that any divisor such that $\gamma = 0$ has zero intersection with r and divisors H and $H-D$ satisfy this condition. Since

$$H.D = H.C = H.r = 0$$

and

$$(H-D).l = (H-D).r = 0,$$

we indeed have each of them contracting something else other than r and so they have to lie on two faces of the cone, thus on the ray being the intersection of these faces.

Let us analyse the face of the cone on which there are divisors Q such that $Q.l = 0$. In particular $H-D$ lies on this face. It is straightforward to see that divisors Q have to be of the form

$$(2\gamma - \beta)H + \beta D + \gamma E$$

for $\beta, \gamma \in \mathbb{R}$ (with some additional conditions on β and γ to maintain nefness which we do not check for the moment). When $\gamma = 0$ we recover $H - D$ or a multiple of thereof and so we can focus on the case when $\gamma \neq 0$. Since we are working on a cone, we can assume $\gamma = 1$ (it cannot be negative as then $Q.r < 0$) and so Q is of the form

$$(2 - \beta)H + \beta D + E.$$

As each such Q has zero intersection with l and a positive intersection with r, we can only hope for it to contract some other curve on D, be it C_0 or something else. We use [22, V. Corollary 4.13] to find the value of β for which $Q_{|D}$ stops being ample on D and denote this divisor L. We know

$$Q_{|D} \sim ah - \Sigma b_i e_i$$

is ample on D if and only if $b_i > 0$, $a > b_i + b_j$, and $2a > \Sigma_{i \neq j} b_i$ for all i, j. Note that

$$Q_{|D} = -\beta H_D + C_0$$

and so for any given C_0 the calculations are straightforward. As we obtain L we see that any linear combination of L and H has to be zero on some set of curves on D (or D itself). In the ideal situation L would be the third ray spanning the Kähler cone of X, and this is the case when $\beta < 2$ as then the intersection with the curves outside of E and D is non negative. This is not always true as evidenced by the Table 3. □

On Figure 1. we present the Kähler cone of X when L is the third spanning ray of X. Note that the further analysis of contractions of X is not impeded should it happen that there are some additional faces of $\bar{K}(X)$. We are mainly interested in the type III contraction that is provided by divisors lying on the face contained between $H - D$ and L.

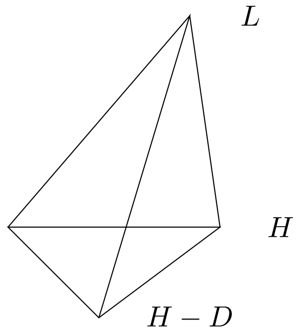

FIGURE 1. Kähler cone of X for L nef

5. Hodge numbers of Calabi-Yau threefolds with double and triple points

Here we generalize formulas by Cynk concering the Hodge numbers of threefolds containing ordinary double and ordinary triple points as the only singularities.

5.1. **Hypersurfaces in weighted projective space.** We base our calculations on [6]. In this work there are explicit results given for calculating Hodge numbers of resolutions of singularities of threefolds containing only ordinary triple points as singularities in four dimensional weighted projective spaces. We want to extend these results to encompass resolutions of normal threefolds - particularly sextics - having ordinary double and triple points as singularities. Note that these conditions imply that threefold X has to miss the singular locus of the weighted projective space $\mathbb{P}$ as otherwise it would inherit its singularities. Our work provides a slight generalization of results from [5] yet requiring a careful retracing of all the steps taken in that work. We conduct the reasoning for a general threefold hypersurface in weighted projective space and later apply it to sextic in $\mathbb{P}(1:1:1:1:2)$ and a quintic threefold in $\mathbb{P}^4$.

For this and the following section we switch the notation for it to be consistent with the usual convention of $\tilde{\mathbb{P}}$ being the blowup of the projective space. Thus, let X be the singular threefold in question, and let $\tilde{X}$ be its resolution. We denote by P_i's the triple points on X and by Q_j's the double points. We use μ_k to denote the number of points of multiplicity k. Then $P = \{P_1, \dots, P_{\mu_3}\}$ and $Q = \{Q_1, \dots, Q_{\mu_2}\}$ constitute singular locus of X which we denote Σ. Write $\pi : \tilde{\mathbb{P}} \to \mathbb{P}$ for the blowup of the weighted projective space in all singular points of X and denote the exceptional divisors of π with $E_i = \pi^{-1}(P_i)$, $D_j = \pi^{-1}(Q_j)$, $E = E_1 + \dots E_{\mu_3}$, $D = D_1, \dots, D_{\mu_2}$. For sets A, B of points on X and non-negative integers m, n let $\mathscr{I}_{mA+nB}$ denote the ideal sheaf of germs of regular functions on $\mathbb{P}$ vanishing at all points of A to order at least m and at all points of B to order at least

n. From the property of blow-up of we have

$$(1) \qquad K_{\tilde{\mathbb{P}}} \sim \pi^* K_{\mathbb{P}} + 3E$$

$$(2) \qquad \tilde{X} \sim \pi^* X - 2D - 3E$$

Since we are working in the weigthed projective space, which is singular - yet still normal in our case - we consider sheaves $\bar{\Omega}^p_{\mathbb{P}}$ and $\bar{\Omega}^p_{\tilde{\mathbb{P}}}$ of germs of p-forms on $\mathbb{P}$ and $\tilde{\mathbb{P}}$. In generality, for normal algebraic variety Y we have that $\bar{\Omega}^p_Y = j_* \Omega_{Reg(Y)}$, where $j : Reg(Y) \to Y$ is the inclusion. Of course for smooth projective space we have $\bar{\Omega}^p_{\mathbb{P}^4} = \Omega^p_{\mathbb{P}^4}$. In our disussion we require the notion of the logarithmic differential forms. For a divisor X that is disjoint from the singular locus of Y we define $\bar{\Omega}^p_Y(\log X)$ to be the sheaf of p-forms ω such that ω and $d\omega$ have at most simple poles along X. As X misses the singular points of Y we see:

$$(3) \qquad \bar{\Omega}^p_Y(\log X)|\mathrm{Reg}(Y) = \Omega^p_{\mathrm{Reg}(Y)}(\log X \cap \mathrm{Reg}(Y))$$

$$(4) \qquad \bar{\Omega}^p_Y(\log X)|(Y \setminus X) = \bar{\Omega}^p_{Y \setminus X}$$

For smooth X we have exact sequences as in ([13, Proposition 2.3]):

$$(5) \qquad 0 \to \bar{\Omega}^p_Y \to \bar{\Omega}^p_Y(\log X) \to \Omega^{p-1}_X \to 0$$

$$(6) \qquad 0 \to \bar{\Omega}^p_Y(\log X)(-X) \to \bar{\Omega}^p_Y \to \Omega^p_X \to 0.$$

Now we recall Proposition 1. of [5] with slightly adapted point (6).

Proposition 5.1. [5, Proposition 1] *Under notation as above we have:*

(1) $\pi_* \mathscr{O}_{\tilde{\mathbb{P}}}(-mE) \cong \mathscr{J}_{m\Sigma}$*, for* $m \geq 0$
(2) $R^i \pi_* \mathscr{O}_{\tilde{\mathbb{P}}}(-mE) = 0$*, for* $i \neq 0, m \geq 0$
(3) $H^i(\mathscr{O}_{\tilde{X}}) = 0$*, for* $i = 1, 2$
(4) $H^i(\Omega^3_{\mathbb{P}}) = 0$*, for* $i \leq 2$.
(5) $H^i(\Omega^4_{\tilde{\mathbb{P}}}(\tilde{X}) \cong H^i(\Omega^4_{\mathbb{P}}(X))$
(6) $H^i(\Omega^4_{\tilde{\mathbb{P}}}(2\tilde{X})) \cong H^i(\Omega^4_{\mathbb{P}}((2X) \otimes \mathscr{J}_{Q+3P})$

Proof. We only need to show (6).We consider

$$K_{\tilde{\mathbb{P}}} + 2\tilde{X} \sim \pi^*(K_{\mathbb{P}} + 2X) - D - 3E$$

and thus from the projection formula we obtain

$$\Omega^4_{\tilde{\mathbb{P}}}(2\tilde{X}) = \Omega^4_{\mathbb{P}}(2X) \otimes \mathscr{J}_{Q+3P}$$

□

The following is [5, Lemma 2]. We quote it with proof for our result to be self-contained.

Lemma 5.2.

$$h^{1.2}(\tilde{X}) = h^0(\Omega^3_{\tilde{X}}(\tilde{X})) - h^0(\bar{\Omega}^3_{\tilde{\mathbb{P}}}(\tilde{X})) + \dim Ker(H^1(\bar{\Omega}^3_{\tilde{\mathbb{P}}}(\tilde{X})) \to H^1(\Omega^3_{\tilde{X}}(\tilde{X})))$$

Proof. From Serre duality it follows that $H^0(\Omega^2_{\tilde{X}}) = H^2(\mathscr{O}_{\tilde{X}}) = 0$. As $\tilde{X}$ is smooth and misses the singular locus of the weighted projective space, the exact sequence from above reads

$$0 \to \bar{\Omega}^3_{\tilde{\mathbb{P}}} \to \bar{\Omega}^3_{\mathbb{P}}(\log \tilde{X}) \to \Omega^2_{\tilde{X}} \to 0$$

and from that we see

$$H^0(\Omega^2_{\tilde{X}}) \cong H^0(\bar{\Omega}^3_{\mathbb{P}}(\log \tilde{X})) = 0$$
$$H^1(\Omega^2_{\tilde{X}}) \cong H^1(\bar{\Omega}^3_{\mathbb{P}}(\log \tilde{X}))$$

Analogously, tensoring the second exact sequence with $\mathscr{O}_{\tilde{\mathbb{P}}}(\tilde{X})$ we get the following:

$$0 \to \bar{\Omega}^3_{\mathbb{P}}(\log \tilde{X}) \to \bar{\Omega}^3_{\tilde{P}}(\tilde{X}) \to \Omega^3_{\tilde{X}}(\tilde{X}) \to 0.$$

Now, the lemma follows from the derived log exact sequence

$$0 \to H^0(\bar{\Omega}_{\tilde{P}}(\tilde{X})) \to H^0(\Omega^3_{\tilde{X}}(\tilde{X})) \to H^1(\bar{\Omega}^3_{\mathbb{P}}(\log \tilde{X})) \to H^1(\bar{\Omega}^3_{\tilde{P}}(\tilde{X})) \to H^1(\Omega^3_{\tilde{X}}(\tilde{X}))$$

□

Lemma 5.3. *The following sequence is exact*

$$H^0\bar{\Omega}^3_{\mathbb{P}}(X) \longrightarrow H^0(\bar{\Omega}^3_{\mathbb{P}}(X) \otimes \mathscr{O}_P) \longrightarrow H^1(\bar{\Omega}^3_{\tilde{\mathbb{P}}}(\tilde{X})) \longrightarrow 0$$

Proof. By [5, Corollary 3] we have

$$\pi^*\bar{\Omega}^3_{\mathbb{P}} \cong \bar{\Omega}^3_{\tilde{\mathbb{P}}}(\log(E+D))(-3(E+D))$$

and

$$\mathscr{O}_{\tilde{\mathbb{P}}}(\tilde{X}) \cong \pi^*(\mathscr{O}_{\mathbb{P}}(X)) \otimes \mathscr{O}_{\tilde{\mathbb{P}}}(-2D-3E).$$

By [13, Property 2.3(c)] we get the following exact sequence

$$0 \longrightarrow \bar{\Omega}^3_{\tilde{\mathbb{P}}}(\log(E+D))(-(E+D)) \longrightarrow \bar{\Omega}^3_{\tilde{\mathbb{P}}} \longrightarrow \bar{\Omega}^3_{(E+D)|(E+D)} \longrightarrow 0$$

which translates into

$$0 \longrightarrow \pi^*\bar{\Omega}^3_{\mathbb{P}}(2(E+D)) \longrightarrow \bar{\Omega}^3_{\tilde{\mathbb{P}}} \longrightarrow \bar{\Omega}^3_{(E+D)|(E+D)} \longrightarrow 0$$

and tensoring with $\mathscr{O}_{\tilde{\mathbb{P}}}(\tilde{X})$ we get

$$0 \longrightarrow \pi^*\bar{\Omega}^3_{\tilde{\mathbb{P}}}(X) \otimes \mathscr{O}_{\tilde{\mathbb{P}}}(-E) \longrightarrow \bar{\Omega}^3_{\tilde{\mathbb{P}}}(X) \longrightarrow \bar{\Omega}^3_E(3) \otimes \bar{\Omega}^3_D(2) \longrightarrow 0$$

Applying the direct image we obtain $\pi_*\bar{\Omega}^3_{\tilde{\mathbb{P}}}(\tilde{X}) = \bar{\Omega}^3_{\mathbb{P}}(X) \otimes \mathscr{J}_P$.Again by [5]

$$H^i\bar{\Omega}^3_{\tilde{\mathbb{P}}}(\tilde{X}) = H^i(\bar{\Omega}^3_{\mathbb{P}}(X) \otimes \mathscr{J}_P)$$

and

$$R^i\pi_*\bar{\Omega}_{\tilde{\mathbb{P}}}(\tilde{X}) = 0$$

for $i > 0$. Long exact sequence coming from

$$0 \longrightarrow \bar{\Omega}^3_{\mathbb{P}}(X) \otimes \mathscr{J}_P \longrightarrow \bar{\Omega}^3_{\mathbb{P}}(X) \longrightarrow \bar{\Omega}^3_{\mathbb{P}}(X) \otimes \mathscr{O}_P \longrightarrow 0$$

gives us

$$H^0(\bar{\Omega}^3_{\mathbb{P}}(X)) \longrightarrow H^0(\bar{\Omega}^3_{\mathbb{P}}(X) \otimes \mathscr{O}_P) \longrightarrow H^1(\bar{\Omega}^3_{\mathbb{P}}(X) \otimes \mathscr{J}_P) \longrightarrow H^1(\bar{\Omega}^3_{\mathbb{P}}(X)) = 0$$

which finishes the proof. □

Lemma 5.4. *We have the following exact sequence*

$$H^0\bar{\Omega}^4_{\mathbb{P}}(2X) \longrightarrow H^0(\bar{\Omega}^4_{\mathbb{P}}(2X) \otimes \mathscr{O}_Q \otimes \mathscr{O}_{3P}) \longrightarrow H^1\bar{\Omega}^3_{\mathbb{P}}(\tilde{X}) \longrightarrow 0$$

Proof. By the adjunction formula $K_{\tilde{X}} = (K_{\tilde{\mathbb{P}}} + \tilde{X})_{|\tilde{X}}$ we get $\bar{\Omega}^3_{\tilde{X}}(\tilde{X}) \cong \bar{\Omega}^4_{\tilde{\mathbb{P}}}(2\tilde{X})|\tilde{X}$. This means that we can translate the short exact sequence

$$0 \longrightarrow \bar{\Omega}^4_{\tilde{\mathbb{P}}}(\tilde{X}) \longrightarrow \bar{\Omega}^4_{\tilde{\mathbb{P}}}(2\tilde{X}) \longrightarrow \bar{\Omega}^4_{\tilde{\mathbb{P}}}(2\tilde{X}) \otimes \mathscr{O}_{\tilde{X}} \longrightarrow 0$$

into

$$0 \longrightarrow \bar{\Omega}^4_{\tilde{\mathbb{P}}}(\tilde{X}) \longrightarrow \bar{\Omega}^4_{\tilde{\mathbb{P}}}(2\tilde{X}) \longrightarrow \bar{\Omega}^3_{\tilde{X}}(\tilde{X}) \longrightarrow 0.$$

From that we obtain the long exact sequence with

$$H^1\bar{\Omega}^4_{\tilde{\mathbb{P}}}(\tilde{X}) \longrightarrow H^1(\bar{\Omega}^4_{\tilde{\mathbb{P}}}(2\tilde{X})) \longrightarrow H^1\bar{\Omega}^3_{\tilde{X}}(\tilde{X}) \longrightarrow H^2\bar{\Omega}^4_{\tilde{\mathbb{P}}}(\tilde{X}) = 0$$

and so

$$H^1\bar{\Omega}^3_{\tilde{X}}(\tilde{X}) \cong H^1(\bar{\Omega}^4_{\tilde{\mathbb{P}}^4}(2\tilde{X}) \cong H^1\bar{\Omega}^4_{\mathbb{P}}((2X) \otimes \mathscr{J}_{Q+3P}).$$

using (6) from the Proposition 5.1 for the last isomorphism. As we have the exact sequence

$$0 \longrightarrow \bar{\Omega}^4_{\mathbb{P}}((2\tilde{X}) \otimes \mathscr{J}_{Q+3P}) \longrightarrow \bar{\Omega}^4_{\mathbb{P}}(2X) \longrightarrow \bar{\Omega}^4_{\mathbb{P}}((2X) \otimes \mathscr{O}_Q \otimes \mathscr{O}_{3P}) \longrightarrow 0$$

the derived one finishes the proof. □

We have $S := \oplus_{d=0}^{\infty} S^d = \mathbb{C}[X_0, \ldots, X_4]$ - the ring of polynomials graded by $\deg(X_i) = w_i$. For a homogeneous ideal $I \subset S$ we write $I^{(d)} := I \cap S^d$, the degree d graded summand. In what follows we need the notion of the equisingular ideal of X, that is

$$I_{eq} := \bigcap_{i=1}^{\mu_3} (m_i^3 + \mathrm{Jac}F) \cap \bigcap_{j=1}^{\mu_2} (m_j + \mathrm{Jac}F).$$

Theorem 5.5. *With notation as above we have*

$$h^{1,1}(\tilde{X}) = \dim(I_{eq}^{(2d-|w|)}) - \dim S^{2d-|w|} + 12\mu_3 + \mu_2 + 1$$
$$h^{1,2}(\tilde{X}) = \dim(I_{eq}^{(2d-|w|)}) - \Sigma_{i=0}^{k} \dim S^{d+w_i-|w|}.$$

Proof. Consider the following diagram with exact rows:

$$
\begin{array}{ccccccc}
\bigoplus_i S^{d+w_i-|w|}/S^{d-|w|} & & \mathbb{C}^{4\mu_3} & & & & \\
\downarrow \cong & & \downarrow \cong & & & & \\
H^0(\bar{\Omega}^3_{\mathbb{P}}(X)) & \xrightarrow{\theta} & H^0(\bar{\Omega}^3_{\mathbb{P}}(X) \otimes \mathscr{O}_P) & \xrightarrow{\alpha} & H^1(\bar{\Omega}^3_{\tilde{\mathbb{P}}^4}(\tilde{X})) & \longrightarrow & 0 \\
\downarrow \xi & & \downarrow \beta & & \downarrow \phi & & \\
H^0(\bar{\Omega}^4_{\mathbb{P}}(2X)) & \xrightarrow{\eta} & H^0(\bar{\Omega}^4_{\mathbb{P}}(2X) \otimes \mathscr{O}_Q \otimes \mathscr{O}_{3P}) & \xrightarrow{\gamma} & H^1(\Omega^3_{\tilde{X}}(\tilde{X})) & \longrightarrow & 0 \\
\cong \uparrow & & \cong \uparrow & & & & \\
S^{2d-|w|} & & \mathbb{C}^{15\mu_3+\mu_2} & & & &
\end{array}
$$

To make this proof self-contained we recall the description of the maps in this diagram from [4]. We denote by K_j the contraction with the vector field $\frac{\partial}{\partial X_j}$ and by Ω the 4-form $\Sigma_{i=0}^4(-1)^i w_i X_j dX_0 \wedge \cdots \wedge d\hat{X}_i \wedge \cdots \wedge dX_4$. The vertical isomorphisms in the first column are as follows:

$$\bigoplus_{i=0}^{k} S^{d+w_i-|w|} \ni (A_0,\ldots,A_4) \mapsto \Sigma_{i=0}^4 \frac{A_i}{F} K_i \Omega \in H^0(\bar{\Omega}^3_{\mathbb{P}}(X))$$

where we have the inclusion $S^{d-|w|} \ni A \mapsto (w_0X_0A,\ldots,w_4X_4A) \in \oplus_{i=0}^k S^{d+w_i-|w|})$ and

$$S^{2d-|w|} \ni A \mapsto \frac{A}{F^2}\Omega \in H^0(\Omega^4_{\mathbb{P}}(2X)).$$

With respect to these isomorphisms, we can think of θ as assigning the values at $P_1,\ldots,P_{\mu_3}$ to quintuples $A_0,\ldots,A_4$ and η as assigning 3-jets at $P_1,\ldots,P_{\mu_3}$ and values at $Q_1,\ldots,Q_{\mu_2}$ to degree $2d-|w|$ homogenous polynomials. Also, ξ is the exterior derivative $\omega \mapsto d\omega$ and thus:

$$d(\Sigma_{i=0}^4 \frac{A_i}{F} K_i \Omega) = \frac{1}{F^2}\Sigma_{i=0}^4 (F\frac{\partial A_i}{\partial X_i} - A_i \frac{\partial F}{\partial X_i})\Omega.$$

Finally, $\beta(A_0^1,\ldots,A_4^1,\ldots,A_0^{\mu_3},\ldots,A_4^{\mu_3})$ is given by 3-jets of $\Sigma_{i=0}^4 A_i^k(P_k)\frac{\partial F}{\partial X_i}$ at $P_1\ldots,P_{\mu_3}$. This last map is injective as F has ordinary triple points at P_i's which means that partial derivatives of F are linearly independent modulo m_i^3 where m_i is the maximal ideal of point P_i. Thus $\dim \mathrm{Im}(\beta) = 4\mu_3$.

We are to calculate $\dim \mathrm{Ker}\Phi$. Note that $h^1(\Omega^3_{\tilde{X}}(\tilde{X})) = h^1(\Omega^3_{\tilde{\mathbb{P}}^4}(\tilde{X})) - \dim \mathrm{Ker}\Phi + \dim \mathrm{Coker}\Phi$. On the other hand for any $b \in H^0(\Omega_{\mathbb{P}}(X) \otimes \mathscr{O}_Q)$ we see that $(\Phi \circ \alpha)(b) = 0 \iff$

$(\gamma\circ\beta)(b)=0$. As β is injective we can, abusing notation, write $b\in H^0(\Omega_{\mathbb{P}}(2X)\otimes\mathscr{O}_Q\otimes\mathscr{O}_{3P})$. But then $\gamma(b)=0 \iff b\in \mathrm{Im}(\eta)$. We have obtained the following exact sequence

$$0\to \mathrm{Im}\beta\cap\mathrm{Im}\eta\to\mathrm{Im}\beta\to H^1(\Omega^3_X(X))\to \mathrm{Coker}(\gamma\circ\beta)\to 0$$

From that $h^1(\Omega^3_X(X)) = \dim\mathrm{Im}(\beta)-\dim(\mathrm{Im}(\beta)\cap\mathrm{Im}(\eta))+\dim\mathrm{Coker}(\beta\circ\gamma)$. But $\dim\mathrm{Coker}\Phi=\dim\mathrm{Coker}(\beta\circ\gamma)$ because it is the dimension of the component of $H^1(\Omega^3_{\tilde{X}}(\tilde{X}))$ that is exactly not in the image of Φ and since α is surjective and β injective this is the same as the component not in the image of $\beta\circ\gamma$. Putting the equalities together we obtain

$$\dim(\mathrm{Ker}\Phi)=h^1(\Omega^3_{\tilde{\mathbb{P}}^4}(\tilde{X}))-\dim(\mathrm{Im}\beta)+\dim(\mathrm{Im}(\beta)\cap\mathrm{Im}(\eta))$$

By [5] Lemma 2. we have the following formula for $h^{1,2}(\tilde{X})$:

$$h^{1,2}(\tilde{X})=h^0(\Omega^3_{\tilde{X}}(\tilde{X}))-h^0(\Omega^3_{\tilde{\mathbb{P}}}(\tilde{X}))+\dim\mathrm{Ker}(H^1\Omega^3_{\tilde{\mathbb{P}}}(\tilde{X})\to H^1\Omega^3_{\tilde{X}}(\tilde{X}))$$

and putting it together with the above result we get

$$h^{1,2}(\tilde{X})=h^0(\Omega^3_{\tilde{X}}(\tilde{X}))-h^0(\Omega^3_{\tilde{\mathbb{P}}}(\tilde{X}))+h^1(\Omega^3_{\tilde{\mathbb{P}}^4}(\tilde{X}))-\dim(\mathrm{Im}\beta)+\dim(\mathrm{Im}(\beta)\cap\mathrm{Im}(\eta))$$

In the calculations below we will use the following exact sequences:

$$0\to\Omega^3_{\mathbb{P}^4}(\bar{X})\otimes\mathscr{J}_Q\to\Omega^3_{\mathbb{P}^4}(\bar{X})\to\Omega^3_{\mathbb{P}^4}(\bar{X})\otimes\mathscr{O}_Q\to 0$$

$$0\to\Omega^4_{\tilde{\mathbb{P}}^4}(X)\to\Omega^4_{\tilde{\mathbb{P}}^4}(2X)\to\Omega^4_{\tilde{\mathbb{P}}^4}(2X)_{|X}\to 0$$

where $\Omega^4_{\tilde{\mathbb{P}}^4}(2X)_{|X}\cong\Omega^3_X(X)$ by adjunction. As $h^1(\Omega^3_{\mathbb{P}^4}(\bar{X}))=0$ from the first sequence we obtain

$$h^1(\Omega^3_{\mathbb{P}^4}(\bar{X})\otimes\mathscr{J}_Q)-h^0(\Omega^3_{\mathbb{P}^4}(\bar{X})\otimes\mathscr{J}_Q)=h^0(\Omega^3_{\mathbb{P}^4}(\bar{X})\otimes\mathscr{O}_Q)-h^0(\Omega^3_{\tilde{\mathbb{P}}^4}(X)).$$

From the second one we have

$$h^0(\Omega^3_X(X))=h^0(\Omega^4_{\tilde{\mathbb{P}}^4}(2X))-h^0(\Omega^4_{\tilde{\mathbb{P}}^4}(X))=h^0(\Omega^4_{\mathbb{P}^4}(2\bar{X})\otimes\mathscr{J}_{3P+Q})-h^0(\Omega^4_{\tilde{\mathbb{P}}^4}(X)).$$

Recall also $h^i(\Omega^3_{\tilde{\mathbb{P}}^4}(X)) = h^i(\Omega^3_{\mathbb{P}^4}(\bar{X}) \otimes \mathscr{I}_Q)$. By the above results we have

$$h^0(\Omega^3_{\tilde{X}}(\tilde{X})) - h^0(\Omega^3_{\tilde{\mathbb{P}}^4}(\tilde{X})) + h^1(\Omega^3_{\tilde{\mathbb{P}}^4}(\tilde{X})) =$$

$$h^0(\Omega^4_{\mathbb{P}}(2X) \otimes \mathscr{I}_{3P+Q}) - h^0(\Omega^4_{\mathbb{P}^4}(X)) - h^0(\Omega^3_{\tilde{\mathbb{P}}^4}(\tilde{X}) \otimes \mathscr{I}_P) + h^1(\Omega^3_{\tilde{\mathbb{P}}^4}(\tilde{X}) \otimes \mathscr{I}_P) =$$

$$h^0(\Omega^4_{\mathbb{P}^4}(2X) \otimes \mathscr{I}_{3P+Q}) - h^0(\Omega^4_{\mathbb{P}^4}(X)) + h^0(\Omega^3_{\mathbb{P}^4}(X) \otimes \mathscr{O}_Q) - h^0(\Omega^3_{\mathbb{P}^4}(X))$$

From the discussion above we obtain that $\mathrm{Im}(\beta) \cap \mathrm{Im}(\eta) = (I_{eq}/(\bigcap_{i=1}^{\mu_3} m_i^3 \cap \bigcap_{j=1}^{\mu_2} m_j))^{(2d-|w|)}$. Now

$$h^0(\Omega^4_{\mathbb{P}^4}(2X) \otimes \mathscr{I}_{3P+Q}) = \dim((\bigcap_{i=1}^{\mu_3} m_i^3 \cap \bigcap_{j=1}^{\mu_2} m_j)^{(2d-|w|)})$$

$$h^0(\Omega^4_{\mathbb{P}^4}(X)) = \binom{d-|w|+4}{4}$$

$$h^0(\Omega^3_{\mathbb{P}^4}(X)) = \Sigma_i \binom{d+w_i-|w|+4}{4} + \binom{d-|w|+4}{4}$$

$$h^0(\Omega^3_{\mathbb{P}^4}(X) \otimes \mathscr{O}_Q) = 4\mu_3$$

and so

$h^{1,2}(\tilde{X}) =$

$$\dim(I_{eq})^{(2d-|w|)} - \binom{d-|w|+4}{4} + 4\mu_3 - \Sigma_i \binom{d+w_i-|w|+4}{4} + \binom{d-|w|+4}{4} - 4\mu_3 =$$

$$\dim(I_{eq})^{(2d-|w|)} - \Sigma_{i=0}^4 \dim S^{d+w_i-|w|}$$

We keep following [4]. We remark that a general hypersurface X_{sm} of degree $d = \deg X$ misses the singular locus of $\mathbb{P}$ and is smooth as X missing $Sing(\mathbb{P})$ implies that the weights $w_0 \dots, w_4$ are pairwise coprime and w_i divides d for all i. Thus by [10, Theorem 3.2.4 and Theorem 4.3.2]

$$e(X_{sm}) = 4 - 2\dim S^{2d-|w|} + 2\Sigma_{i=0}^4 \dim S^{d+w_i-|w|} - 2\dim S^{d-|w|}.$$

The Milnor number of an ordinary triple point is 16 and the resolution replaces it with the cubic surface with Euler number 9; similarily, small resolution of a double point with Milnor number 1 replaces it with a $\mathbb{P}^1$ with Euler number 2. Thus:

$$e(\tilde{X}) = 4 - 2\dim S^{2d-|w|} + 2\Sigma_{i=0}^4 \dim S^{d+w_i-|w|} - 2\dim S^{d-|w|} + 24\mu_3 + 2\mu_2.$$

Now, using the fact that $e(\tilde{X}) = 2(h^{1,1}(\tilde{X}) - h^{1,2}(\tilde{X}) - h^{0,3}(\tilde{X}) + 1)$ (analogously for X_{sm}) and the fact that $H^{0,3}(\tilde{X}) = S^{d-|w|}$ as the resolution of X is crepant we obtain

$$\begin{aligned} h^{1,1}(\tilde{X}) = &\frac{e(X_{sm})}{2} + 12\mu_3 + 2\mu_2 + h^{1,2}(\tilde{X}) + h^{0,3}(\tilde{X}) - 1 = \\ &2 - \dim S^{2d-|w|} + \Sigma_{i=0}^4 \dim S^{d+w_i-|w|} - \dim S^{d-|w|} + 12\mu_3 + \mu_2 + \\ &\dim(I_{eq})^{(2d-|w|)} - \Sigma_{i=0}^4 \dim S^{d+w_i-|w|} + S^{d-|w|} - 1 = \\ &\dim(I_{eq}^{(2d-|w|)}) - \dim S^{2d-|w|} + 12\mu_3 + \mu_2 + 1 \end{aligned}$$

as desired. □

Recall that the *defect* of the hypersurface X is the difference between the actual and the expected dimension of the ideal $I_{eq}^{(2d-|w|)}$. As the codimension of the ideal $(m_i^3 + \text{Jac}F)$ is 11 and the codimension of $(m_j + \text{Jac}F)$ is 1 we have

$$\delta = \dim I_{eq}^{(2d-|w|)} - (\dim S^{2d-|w|} - 11\mu_3 - \mu_2)$$

and thus we can write our results in more approachable form:

Corollary 5.6. *With notation as before we have*

$$h^{1,2}(\tilde{X}) = \dim S^{2d-|w|} - \Sigma_{i=0}^4 \dim S^{d+w_i-|w|} - 11\mu_3 - \mu_2 + \delta$$

and

$$h^{1,1}(\tilde{X}) = 1 + \mu_3 + \delta.$$

Note that the only difference between these results and results from [5] is the number of double points μ_2 both appearing in the expression for $h^{1,2}(\tilde{X})$ and "hidden" in the defect δ.

We can now easily apply these results to the quintic threefold and to the sextic in the projective weighted space. Taking also into account the remaining Hodge numbers we obtain:

Corollary 5.7. *For $\tilde{X}$, a resolution of singularities of a quintic threefold $X \subset \mathbb{P}^4$ with only ordinary triple points and ordinary double points as singularities the following holds:*

(1) $h^{0,0}(\tilde{X}) = h^{3,3}(\tilde{X}) = h^{3,0}(\tilde{X}) = 1$

(2) $h^{1,0}(\tilde{X}) = 0$

(3) $h^{1,1}(\tilde{X}) = 1 + \mu_3 + \delta$

(4) $h^{1,2}(\tilde{X}) = 101 - 11\mu_3 - \mu_2 + \delta.$

Corollary 5.8. *For $\tilde{X}$, a resolution of singularities of a sextic threefold $X \subset \mathbb{P}(1:1:1:1:2)$ with only ordinary triple points and ordinary double points as singularities the following holds:*

(1) $h^{0,0}(\tilde{X}) = h^{3,3}(\tilde{X}) = h^{3,0}(\tilde{X}) = 1$

(2) $h^{1,0}(\tilde{X}) = 0$

(3) $h^{1,1}(\tilde{X}) = 1 + \mu_3 + \delta$

(4) $h^{1,2}(\tilde{X}) = 175 - 11\mu_3 - \mu_2 + \delta.$

5.2. **Complete intersection threefolds.** We move on to calculate Hodge numbers of the complete intersection threefolds that we consider. Once again we will be basing our results on work by Cynk [5]. We consider his results regarding nodal complete intersection threefolds and provide a generalization concerning complete intersection threefolds which can have only ordinary double and triple points as singularities. For simplicity we limit ourselves to the complete intersection of two hypersurfaces in $\mathbb{P}^5$ although these results could be generalized to higher dimensions. Let X be the complete intersection $H_1 \cap H_2 \subset \mathbb{P}^5$ of 2 hypersurfaces in $\mathbb{P}^5$. We require the hypersurfaces to be smooth. Also write $Y = H_1$. See that now X is the hypersurface in the smooth projective fourfold Y.

Again we denote by P_i's the triple points on X and by Q_j's the double points. We use μ_k to denote the number of points of multiplicity k. Then $P = \{P_1, \dots, P_{\mu_3}\}$ and $Q = \{Q_1, \dots, Q_{\mu_2}\}$ constitute singular locus of X which we denote Σ. Write $\pi : \tilde{Y} \to Y$ for the blowup of Y in all singular points of X and denote the exceptional divisors of π with $E_i = \pi^{-1}(P_i)$, $D_j = \pi^{-1}(Q_j)$, $E = E_1 + \dots E_{\mu_3}$, $D = D_1, \dots, D_{\mu_2}$. For sets A, B of points on X and non-negative integers m, n let $\mathscr{I}_{mA+nB}$ denote the ideal sheaf of germs of regular functions on $\mathbb{P}$ vanishing at all points of A to order at least m and at all points of B to order at least n. This time let I denote the module generated by rows of the Jacobian matrix $\mathrm{Jac}(F_1, F_2)$. The following is the analogue of the Proposition 5.1 adapted to our case and thus we present it without proof.

Proposition 5.9. *We have*

(1) $\pi_* \mathscr{O}_{\tilde{Y}}(-mE) \cong \mathscr{I}_{m\Sigma}$, *for* $m \geq 0$,
(2) $R^i \pi_* \mathscr{O}_{\tilde{Y}}(-mE) = 0$, *for* $i \neq 0$, $m \geq 0$,
(3) $H^i(\mathscr{O}_{\tilde{X}}) = 0$, *for* $i = 1, 2$,
(4) $H^0(\Omega^4_{\tilde{Y}}(\tilde{X}) \cong H^0(\Omega^4_Y(X))$,
(5) $H^i(\Omega^4_{\tilde{Y}}(\tilde{X})) = 0$, *for* $i > 0$,
(6) $H^i(\Omega^4_{\tilde{Y}}(2\tilde{X}) \cong H^i(\Omega^4_Y(2X) \otimes \mathscr{I}_{3P+Q})$

We also see the analogue of the lemmas from the hypersuface case, we adapt them here without proof.

Lemma 5.10.

$$h^{1.2}(\tilde{X}) = h^0(\Omega^3_{\tilde{X}}(\tilde{X})) - h^0(\bar{\Omega}^3_{\mathbb{P}}(\tilde{X})) + \dim Ker(H^1(\bar{\Omega}^3_{\mathbb{P}}(\tilde{X})) \to H^1(\Omega^3_{\tilde{X}}(\tilde{X})))$$

Lemma 5.11. *We have the following exact sequence*

$$H^0\Omega^4_Y(2X) \longrightarrow H^0(\Omega^4_Y(2X) \otimes \mathscr{O}_Q \otimes \mathscr{O}_{3P}) \longrightarrow H^1\Omega^3_Y(\tilde{X}) \longrightarrow 0$$

Lemma 5.12. *The following sequence is exact:*

$$0 \longrightarrow H^1(\Omega^3_Y) \longrightarrow H^1(\Omega^3_{\tilde{Y}}(\log \tilde{X}) \longrightarrow H^1(\Omega^2_{\tilde{X}}) \longrightarrow 0$$

Lemma 5.13. *The following sequence is exact:*

$$\begin{aligned} 0 \longrightarrow H^0(\Omega^3_Y(X)) \longrightarrow H^0(\Omega^3_{\tilde{X}}(\tilde{X})) \longrightarrow H^1(\Omega^3_{\tilde{Y}}(\log \tilde{X})) \\ \longrightarrow H^1(\Omega^3_Y(X)) \longrightarrow H^1(\Omega^3_{\tilde{X}}(\tilde{X})). \end{aligned}$$

By 5.12 we have:

$$h^1(\Omega^2_{\tilde{X}}) = h^1(\Omega^3_{\tilde{Y}}(\log \tilde{X})) - h^1(\Omega^3_Y).$$

By 5.13 we have:

$$h^1(\Omega^3_{\tilde{Y}}(\log \tilde{X})) = h^0(\Omega^3_{\tilde{X}}(\tilde{X})) - h^0(\Omega^3_Y(X)) + \dim(\mathrm{Ker}(H^1(\Omega^3_Y(X) \longrightarrow H^1(\Omega^3_{\tilde{X}}(\tilde{X})))).$$

By the short exact sequence

$$0 \longrightarrow \Omega^4_{\tilde{Y}}(\tilde{X}) \longrightarrow \Omega^4_{\tilde{Y}}(2\tilde{X})) \longrightarrow \Omega^4_{\tilde{Y}}(2\tilde{X}) \otimes \mathscr{O}_{\tilde{X}} \longrightarrow 0$$

and the adjunction formula

$$\Omega^3_{\tilde{X}}(\tilde{X}) \cong \Omega^4_{\tilde{Y}}(2\tilde{X}) \otimes \mathscr{O}_{\tilde{X}}$$

we obtain

$$h^0(\Omega^4_{\tilde{Y}}(2\tilde{X})) = h^0(\Omega^4_{\tilde{Y}}(\tilde{X})) + h^0(\Omega^3_{\tilde{X}}(\tilde{X}).$$

We get:

$$h^1(\Omega^2_{\tilde{X}}) =$$
$$h^0(\Omega^4_{\tilde{Y}}(2\tilde{X})) - h^0(\Omega^4_{\tilde{Y}}(\tilde{X})) - h^1(\Omega^3_{Y}) - h^0(\Omega^3_{Y}(X)) + \dim(\mathrm{Ker}(H^1(\Omega^3_{Y}(X) \to H^1(\Omega^3_{\tilde{X}}(\tilde{X}))))$$

We want to show the following.

Theorem 5.14. *With notation as above we have*

$$h^{1,1}(\tilde{X}) = \dim(I \cap \mathscr{J}^{2d-2r-3}_{3P+Q}) - \dim I^{2d-2r-3} + 12\mu_3 + \mu_2 + 1$$
$$h^{1,2}(\tilde{X}) = h^{1,2}(X_{smooth}) + \dim(I \cap \mathscr{J}^{2d-2r-3}_{3P+Q}) - \dim I^{2d-2r-3}.$$

Proof. As before we consider the diagram with exact rows:

$$\begin{array}{ccccccc}
 & & S^{d+d_1-6} & \xrightarrow{\quad\alpha\quad} & H^1(\Omega^3_Y(X)) & \longrightarrow & 0 \\
 & & \downarrow \beta & & \downarrow \phi & & \\
H^0(\Omega^4_Y(2X)) & \xrightarrow{\xi} & H^0(\Omega^4_Y(2X) \otimes \mathscr{O}_Q \otimes \mathscr{O}_{3P}) & \xrightarrow{\gamma} & H^1(\Omega^3_{\tilde{X}}(\tilde{X})) & \longrightarrow & 0 \\
\eta \uparrow & \nearrow \theta & \cong \uparrow & & & & \\
S^{d+d_2-6} & & \mathbb{C}^{15\mu_3+\mu_2} & & & &
\end{array}$$

We discuss the vertical identification in the first column. To a polynomial $A \in S^{d+d_2-6}$ we associate a form $\frac{A}{F_1 F_2^2}\Omega$ where $\Omega = \Sigma_{i=0}^{5} X_i dX_0 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge dX_5$. Again let K_i denote the contraction with the vector field X_i. We can treat η as assigning A its value at $Q_1, \ldots, Q_{\mu_2}$ and its 3-jets at $P_1 \ldots P_{\mu_3}$ as before. Let $B \in S^{d+d_1-6}$. Then, similarly to [5] $\beta(B)$ is given by evaluating at Q and assigning 3-jets at P to $\Sigma_{i=0}^{5} B \frac{\partial F_2}{\partial X_i}$. We obtain

$$\mathrm{Im}(\beta) + \mathrm{Im}(\theta) \cong (I \otimes \mathscr{O}_{Q+3P})^{2d-2r-3} \cong I^{2d-2r-r}/(I \cap \mathscr{J}_{Q+3P})^{2d-2r-3}.$$

From 5.9 we have $h^0(\Omega^4_{\tilde{Y}}(\tilde{2}X)) = h^0(\Omega^4_Y(2X) \otimes \mathscr{J}_{3P+Q}) = \dim \mathrm{Ker}(\xi) = h^0(\Omega^4_Y(2X)) - \dim \mathrm{Im}\xi$. Also $\dim \mathrm{Ker}\Phi = h^1(\Omega^3_Y(X)) - \dim \mathrm{Im}\Phi$. Since α and β are epimorphisms we have $\mathrm{Im}\xi = \mathrm{Im}\theta$ and $\mathrm{Im}\phi = \mathrm{Im}(\gamma \circ \beta)$. Now see

$$\dim(\mathrm{Im}\theta + \mathrm{Im}\beta) = \dim \mathrm{Im}\theta + \dim \mathrm{Im}\beta - \dim(\mathrm{Im}\theta \cap \mathrm{Im}\beta) =$$

$$\dim \mathrm{Im}\theta + \dim \mathrm{Im}\beta - \dim(\mathrm{Im}\beta \cap \mathrm{Ker}\gamma) =$$

$$\dim \mathrm{Im}\theta + \dim \mathrm{Im}\beta - \dim(\mathrm{Ker}(\gamma|_{\mathrm{Im}\theta)}) =$$

$$\dim \mathrm{Im}\theta + \dim \mathrm{Im}(\gamma|_{\mathrm{Im}\beta}) =$$

$$\dim \mathrm{Im}\theta + \dim \mathrm{Im}(\gamma \circ \beta) = \dim \mathrm{Im}\theta + \dim \mathrm{Im}\phi = \dim \mathrm{Im}\xi + \dim \mathrm{Im}\phi$$

Putting these into

$$h^1(\Omega^2_{\tilde{X}}) =$$

$$h^0(\Omega^4_{\tilde{Y}}(2\tilde{X})) - h^0(\Omega^4_{\tilde{Y}}(\tilde{X})) - h^1(\Omega^3_{Y}) - h^0(\Omega^3_{Y}(X)) + \dim(\mathrm{Ker}(H^1(\Omega^3_{Y}(X) \to H^1(\Omega^3_{\tilde{X}}(\tilde{X}))))$$

we get

$$h^1(\Omega^2_{\tilde{X}}) = h^0(\Omega^4_{Y}(2X) - h^0(\Omega^4_{Y}(X)) - h^1(\Omega^3_{Y})) - h^0(\Omega^3_{Y}(X)) + h^1(\Omega^3_{Y}(X)) - \dim(\mathrm{Im}\beta + \mathrm{Im}\theta).$$

As for the smooth complete intersection X_{smooth} of the same multi-degrees the computations yield $h^1(\Omega^2_X) = h^0(\Omega^4_Y(2X) - h^0(\Omega^4_Y(X)) - h^1(\Omega^3_Y)) - h^0(\Omega^3_Y(X)) + h^1(\Omega^3_Y(X))$ we conclude

$$h^{1,2}(\tilde{X}) = h^{1,2}(X_{smooth}) - \mu_2 - 11\mu_3 + \delta$$

with the defect

$$\delta = \mu_2 + 11\mu_3 - (\dim I^{2d-2r-r} - \dim(I \cap \mathscr{J}_{Q+3P})^{2d-2r-3}).$$

By similar reasoning as in the case of hypersurfaces in weighted projective spaces we obtain the formula for $h^{1,1}(\tilde{X})$ concluding the proof. □

As before we have immediate application of the formula to the threefolds we are particularly interested in.

Corollary 5.15. *For $\tilde{X}_{2,4}$, a resolution of singularities of a complete intersection threefold $X_{2,4} \subset \mathbb{P}^5$ with only ordinary triple points and ordinary double points as singularities the following holds:*

(1) $h^{0,0}(\tilde{X}) = h^{3,3}(\tilde{X}) = h^{3,0}(\tilde{X}) = 1$

(2) $h^{1,0}(\tilde{X}) = 0$

(3) $h^{1,1}(\tilde{X}) = 1 + \mu_3 + \delta$

(4) $h^{1,2}(\tilde{X}) = 89 - 11\mu_3 - \mu_2 + \delta.$

Corollary 5.16. *For $\tilde{X}_{3,3}$, a resolution of singularities of a complete intersection threefold $X_{3,3} \subset \mathbb{P}^5$ with only ordinary triple points and ordinary double points as singularities the following holds:*

(1) $h^{0,0}(\tilde{X}) = h^{3,3}(\tilde{X}) = h^{3,0}(\tilde{X}) = 1$

(2) $h^{1,0}(\tilde{X}) = 0$

(3) $h^{1,1}(\tilde{X}) = 1 + \mu_3 + \delta$

(4) $h^{1,2}(\tilde{X}) = 73 - 11\mu_3 - \mu_2 + \delta.$

5.3. **Hodge numbers of Calabi-Yau threefolds after a geometric transition.** We return to the notation from before meaning X is again a smooth Calabi-Yau threefold.

Proposition 5.17. *Let X be a smooth Calabi-Yau threefold containing a smooth surface E ruled over a smooth curve C of genus $g(C) > 1$, let $\pi : X \to Y$ be a primitive type III contraction and let $\tilde{Y}$ be the smooth Calabi-Yau obtained by deforming Y. Then*

(1) $h^{1,1}(\tilde{Y}) = h^{1,1}(X) - 1$
(2) $h^{1,2}(\tilde{Y}) = h^{1,2}(X) + 2g(C) - 3.$

Proof. Let $\pi : \mathscr{X} \to \Delta$ be the Kuranishi family for $X = X_0$ where Δ is a polydisc in $H^1(T_X)$ which we identify with the space of deformations of X. By 2.20 the primitive type III contraction $\pi_0 : X \to Y$ deforms to a primitive type I contraction $\pi_t : X_t \to Y_t$ of $2g(C) - 2$ curves of X_t. As E is a smooth surface ruled over a smooth curve C we see from the discussion in the proof of [42, Proposition 4.2] that the fibers are being contracted to A_1 singularities and thus indeed this process yields a conifold transition $T(X_t, Y_t, \tilde{Y})$. Note that we have used the assumption that E is a ruled surface over a smooth curve with $g(C) > 1$. In general a type III contraction deforming to type I contractions may yield some worse singularities, for example if E has double fibers.

By the discussion in [42, p. 562], we can identify groups $H^2(X_t, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$ in a family $\pi : \mathscr{X} \to \Delta$ for some polydisc $\Delta \in H^1(T_X)$. We perform the contractions $\pi_t : X_t \to Y_t$ over Δ and then the smoothing of the image of the type I contraction. We know from [17, Proposition 3.1] and [29, Remark 12.2.1.4.2] that the Picard number of Y_t is constant for $t \in \Delta$ (shrinking Δ if necessary) and thus

$$h^{1,1}(\tilde{Y}) = h^{1,1}(Y_t) = h^{1,1}(X_t) - 1 = h^{1,1}(X) - 1.$$

The final equality follows easily from [39, Theorem 3.2]. It gives that for a conifold transition $T(X_t, Y, \tilde{Y})$ we have

$$h^{1,1}(\tilde{Y}) = h^{1,1}(X_t) - k$$

$$h^{1,2}(\tilde{Y}) = h^{1,2}(X_t) + c$$

where $k + c = |\text{Sing}(Y)|$. Since $|\text{Sing}(Y)| = 2g(C) - 2$ we obtain $c = 2g(C) - 3$ as $k = 1$, thus concluding the proof. □

Even though 2.20 holds for $g = 1$ as well our formulas don't apply then. In this case type III contraction no longer deforms to type I contractions and we cannot discuss the change in Hodge numbers as above.

6. Examples of Calabi-Yau threefolds from type III contractions

6.1. **Calabi-Yau threefolds from quintic threefolds.** Before we move further with the discussion we want to recall one more interesting property of quintic threefolds we work with.

Fact 6.1. *We can look at the geometry of quintic $\bar{X}$ as that of a double octic that is a double covering of $\mathbb{P}^3$ branched over an octic surface (e.g. as in* [8]*). Recall that $\bar{X}$ is described as a zero locus of $F(x,y,z,t,u) = u^2F_3(x,y,z,t) + uF_4(x,y,z,t) + F_5(x,y,z,t)$. We can project X onto $\mathbb{P}^3$ from the point $[0:0:0:0:1]$. Then the branch locus is defined as the vanishing set of the discriminant, namely a surface of degree 8, necessarily of the form $S = V(F_4^2 - 4F_3F_5) \subset \mathbb{P}^3$.*

There are many interesting properties linking the geometry of X with that of S. We show the basic one.

Proposition 6.2. *The singular points of S contain the intersection of $V(F_3)$, $V(F_4)$, $V(F_5)$. In particular, $C \subset Sing(S)$.*

Proof. Let $G := F_4^2 - 4F_3F_5$. A straightforward calculation of derivatives gives us $\frac{\partial G}{\partial \alpha} = 2F_4\frac{\partial F_4}{\partial \alpha} - 4(F_3\frac{\partial F_5}{\partial \alpha} + F_5\frac{\partial F_3}{\partial \alpha})$ for $\alpha \in \{x,y,z,t\}$ and so whenever $F_3 = F_4 = F_5 = 0$ we have $\frac{\partial G}{\partial \alpha} = 0$. This shows $C \subset Sing(S)$. Also the excess points are all contained in the intersection of X_3, X_4, X_5. □

We will not discuss the properties of the double cover any further as this is not the aim of our paper. It is worth noting, though, that some restrictions on the existence of singular octic surfaces translate into restrictions of existence of quintic threefolds with triple point(s). For example one can show that for $m > 1$ quintic threefold with m ordinary triple points is a double cover of $\mathbb{P}^3$ branched over an octic surface with $m-1$ quadruple points. As there are restrictions to the number of isolated quadruple points on an octic [3, Proposition 5.1], this provides interesting results. Note that a generic quintic threefold containing a triple point contains 60 lines, as the intersection of $V(F_3), V(F_4)$ and $V(F_5)$ consists of 60 points. Yet, for $m \geq 10$ a quintic threefold must necessarily contain cones whose vertices are triple points in question as singularites on octic can no longer be isolated forcing $V(F_3), V(F_4)$ and $V(F_5)$ to intersect in a (not necessarily irreducible) curve. This may prove fruitful in further research.

We want to describe the Kähler cone of the constructed threefolds. The formula for the defect is $\delta = \dim I_{eq}^{(5)} - 115 - \mu_2$ and we use it to perform the calculations in Macaulay2 [16]. As expected we obtain $\delta = 1$ for almost all quintics X_i and thus $h^{1,1}(X_i) = 3$ which is also the rank of their Picard groups as discussed previously. The only exception is $X_{3,5}$ for which $\delta = 0$ and $h^{1,1} = 2$. The following Fact is a straightforward application of Theorem 4.6.

Fact 6.3. *For all threefolds obtained by resolution of singularities on quintic threefolds that we consider the closure $\bar{K}$ of the Kähler cone K is contained in the convex hull of three rays. Furthermore, two of the rays are generated by the divisors H and $H - D$ and don't lie on the cubic cone W. The third ray is generated by a divisor L such that $L^3 \neq 0$ for all threefolds but for $X_{2,3}, X_{8B}, X_{11}$ for which $L^3 = 0$.*

In Table 3 we present values β for which $Q_{|D}$ stops being ample, the divisor $L = (2-\beta)H + \beta D + E$ and value L^3 for each of the quintics X. In column $L_{|D}$ we provide the coordinates in the Picard group of D of L.

We are now ready to prove the following statement:

Theorem 6.4. *Threefolds $X_{5(i)}$ constructed in this paper admit*

(1) a type I, a type II, and a type III primitive contractions for $i \in \{3, \{2,3\}, \{3,3\}, \{3,4\}\}$, that is whenever the class of a curve C is a multiple of a class of a hyperplane section of cubic surface,
(2) a type I and two type III primitive contractions for $i \in \{2, 5A, 5B, 8B, 11\}$,
(3) two type I primitive contractions and a type III primitive contraction in other cases.

Proof. This theorem is a consequence of the previous one and of the [42, Fact 1]. Codimension one face spanned by $H - D$ and H corresponds to a type I contraction as it contracts r, face spanned by $H - D$ and L corresponds to a type III contraction as it contracts all fibers l of E and thus contracts E to a curve. The face spanned by L and H yields different contraction depending on the threefold in question. In cases $i \in \{3, \{2, 3\}, \{3, 3\}, \{3, 4\}\}$ we have $L.D = 0$ which means the whole cubic surface is contracted to a point giving a type II contraction. In cases $i \in \{2, 5A, 5B, 8B, 11\}$ we have $L_{|D}$ of class $l - e_1$ or $2l - e_1 - e_2 - e_3 - e_4$. The corresponding complete linear systems are only two-dimensional and thus they provide morphisms of D to a projective line $\mathbb{P}^1$. These are type III contractions of a cubic surface to a curve of genus 0. We do not discuss them in detail as this situation is more complicated than when $g \geq 1$. In the remaining cases we obtain the contraction of some families of curves on D into points, that is type I contractions. □

Note the fact that $L^3 = 0$ for some of the quintics does not affect the contraction corresponding to the faces of the cone limited by L as for all divisors Q on these faces $Q^3 \neq 0$. In addition, take notice that the contraction of X given by the system $|H|$ is exactly the blow-up of a triple point and the curves coming from the small resolution, while system $|H - D|$, that is the system of hyperplane sections passing through D which translates to hyperplane sections passing through the point O on $\bar{X}$, is the system giving a double cover of $\mathbb{P}^3$ discussed earlier. In the following section we take a closer look at the type III contraction.

6.1.1. *Contraction and smoothing.* We restrict our focus to the type III contraction admitted by the quintics X_i. We keep the notation as above. The goal of this section is to prove three theorems regarding the existence of type III contractions.

Theorem 6.5. *For discussed threefolds obtained from quintics $X_{5(i)}$, where $i \notin \{1, 2, \{TC\}, \{4A\}, \{4B\}, \{7A\}\}$, the morphism $\phi_{(|m(2H+E)|)} : X \to Y$ for some $m >> 0$ is a primitive contraction of type III which contracts the ruled surface E to a curve $\tilde{C}$.*

By $\phi_{(|m(2H+E)|)}$ we mean a morphism given by the nef linear system $|m(2H + E)|$ on X_i. We've excluded those quintics for which $2H+E$ has zero, or negative intersection with some curve on D leaving only those for which $(2H+E)_{|D}$ (and thus its positive multiple) is ample.

Proof. First of all we remark that $2H+E$ is indeed contained in the face of the Kähler cone spanned by $H-D$ and L as it is of the form $(2\gamma-\beta)H+\beta D+\gamma E$ for $\beta=0$ and $\gamma=1$ and so $(2H+E).l=0$ for any quintic X_i. Since $|H|$ is the pullback of a hyperplane section of $\bar{X}$ it separates points on X outside of D and $|2H+E|$ separates points outside of D and E. Moreover, restriction of $T \in |2H+E|$ to D belongs to the linear system $|\tilde{C}_i|$ of curves linearly equivalent to some curve $\tilde{C}_i$ on D (different for each X_i). For all quintics we discuss here, the system $|\tilde{C}_i|$ is (very) ample yet we are not sure if $|2H+E|$ maps surjectively onto it and thus whether or not its restriction to D separates points. Yet, following the [42, Fact 1], we know codimension one faces of the Kähler cone correspond to the primitive contractions, and thus some multiple of $|2H+E|$ gives the desired morphism finishing the proof. □

Note $mT \in |m(2H+E)|$ corresponds to the hyperplane section of Y and thus we have that $mT.E$ is actually the degree of the curve $\tilde{C}$. By similar argument we obtain the degree of Y namely $(mT)^3 = m^3(2H+E)^3 = m^3(36+6\deg(C)+4g(C))$. We have strong reasons to believe that $m=1$ should suffice to give the desired morphism, although we haven't been able to show it yet.

We need to deal with the remaining threefolds and their respective type III contractions.

Theorem 6.6. *For threefolds obtained from quintics $X_{5(i)}$ where $i \in \{2, \{TC\}, \{4A\}, \{4B\}, \{7A\}\}$ the morphism $\phi_{(m|3H-D+E|)}: X \to Y$, for some $m >> 0$ is a primitive contraction of type III which contracts the ruled surface E to a curve $\tilde{C}$.*

Proof. We can follow the proof of the previous theorem. As in the preceeding case we note that $T := 3H-D+E$ lies on the face spanned by $H-D$ and $L = 2H+E$. The intersection of T with any curve not contained in D or E is positive as $3H$ intersects any such curve. The divisor is chosen so that it has zero intersection with fiber l of E and in all cases of this theorem $(-D+E)_{|D}$ is an ample curve on D, and thus intersection with any curve contained in D is positive. By [42, Fact 1] we obtain the desired morphism.

□

Theorem 6.7. *For a threefold obtained from quintic $X_{5(1)}$ the morphism $\phi_{(m|4H-2D+E|)}$: $X \to Y$ for some $m >> 0$ is a primitive contraction of type III which contracts the ruled surface E to a curve $\tilde{C}$.*

Proof. The proof follows the same argument as the two before. □

Similarly to the case where $L = m(2H+E)$ we obtain the degrees of Y's and $\tilde{C}$'s. Table 4 summarizes the obtained results.

This brings us to the final result of this section.

Corollary 6.8. *The threefolds X_i for $i \in \{3, 4A, 4B, 5A, 5B, 6, \{2,3\}, 7A, 7B, 7C, 8A, 8B, 9A, 9B, \{3,3\}, 10A, 10B, 11, \{3,4\}\}$ admit a primitive, type III contraction to a threefold Y_i which is smoothable, thus yielding a smooth Calabi-Yau threefold $\tilde{Y}_i$.*

Proof. Immediate from the theorem above and 2.19. □

Remark 6.9. To obtain the Hodge numbers of X for specific cases we need to know the number μ_2 of ordinary double points on $\bar{X}$ or equivalently on $\tilde{X}$. This is actually equal to $c_2(I_{\tilde{E}}/I_{\tilde{E}^2}(5))$ where $I_{\tilde{E}}$ is the defining ideal of the smooth ruled surface $\tilde{E}$ and c_2 denotes the second Chern class of a vector bundle. Note that after blowing up the triple point O of $\bar{X}$ we obtain a singular threefold containing a smooth ruled surface in $\tilde{\mathbb{P}^4}$ - a projective space with a blown-up point. Then the twisted normal bundle $N_{\tilde{E}/\tilde{\mathbb{P}^4}}(5)$ is trivial exactly when a general degree 5 threefold containing $\tilde{E}$ is singular. This translates to the vanishing of a general section of a conormal bundle $N^{\vee}_{\tilde{E}/\tilde{\mathbb{P}^4}}(5) \cong I_{\tilde{E}}/{I_{\tilde{E}}}^2(5)$, and the number of zeroes of such section is $c_2(I_{\tilde{E}}/I^2_{\tilde{E}}(5))$ as the bundle is of rank two. A similar argument in case of complete intersection threefolds can be found in [24, Remark 2.1].

Remark 6.10. It is worth noting that the curve $C_{3,5}$ that is the complete intersection of a cubic and a quintic surface in $\mathbb{P}^3$ also gives rise to a threefold with Picard rank two. As mentioned in the proof of Theorem 4.1 for the threefold obtained by blowing up a triple point of a quintic threefold containing a cone over $C_{3,5}$ we have $E = 3H - 5D$. In this case we have no excess lines and no double points of X lying on the ruled surface E. We can still apply the formulas for $h^{1,1}$ and $h^{1,2}$ to obtain their values for $X_{5(3,5)}$. For a smooth threefold $X_{5(3,5)}$ we have $\delta = 0$, $h^{1,1} = 2$ and $h^{1,2} = 90$. We can also perform the contraction of $E = 3H - 5D$. The linear system $|m(2H + E)| = |m(5H - 5D)|$ for some $m > 0$ gives a birational morphism of $X_{5(3,5)}$ contracting E to a curve C of genus 31. This is the curve of the highest genus that we have managed to obtain as an image of an exceptional divisor through a type III contraction. The image threefold $Y_{5(3,5)}$ is smoothable by the previous results and so we obtain a smooth Calabi-Yau threefold $\tilde{Y}_{5(3,5)}$ of Picard rank 1.

6.2. **Calabi-Yau threefolds from complete intersection threefolds.**

6.2.1. *Complete intersection* $X_{2,4}$. We present the results for complete intersection threefolds $X_{2,4}$ and $X_{3,3}$ without proofs as they are completely analogous to the ones for quintic threefolds.

Theorem 6.11. *Threefolds* $X_{2,4(i)}$ *constructed in this paper admit*

(1) a type I, a type II, and a type III primitive contractions for $i = 6$*, that is whenever the class of a curve* C *is a multiple of a class of a hyperplane section of cubic surface,*
(2) a type I and two type III primitive contractions for $i = 5$*,*
(3) two type I primitive contractions and a type III primitive contraction in case $i = 4$*.*

Theorem 6.12. *For discussed* $X = X_{2,4(5)}$ *or* $X = X_{2,4(6)}$ *the morphism* $\phi_{(|m(2H+E)|)} : X \to Y$ *for some* $m >> 0$ *is a primitive contraction of type III which contracts the ruled surface* E *to a curve* $\tilde{C}$*. For* $X = X_{2,4(4)}$ *the morphism* $\phi_{(|m(3H-D+E)|)} : X \to Y$ *for some* $m >> 0$ *serves the same role.*

Corollary 6.13. *The threefolds $X_{2,4(i)}$ for $i \in \{4, 5, 6\}$ admit a primitive, type III contraction to a threefold Y_i which is smoothable, thus yielding a smooth Calabi-Yau threefold $\tilde{Y}_{2,4(i)}$.*

6.2.2. *Complete intersection $X_{3,3}$.*

Theorem 6.14. *Threefolds $X_{3,3(i)}$ constructed in this paper admit*

(1) a type I, a type II, and a type III primitive contractions for $i \in 3, 6$, that is whenever the class of a curve C is a multiple of a class of a hyperplane section of cubic surface,
(2) a type I and two type III primitive contractions for $i = 5$,
(3) two type I primitive contractions and a type III primitive contraction in case $i = 4$.

Corollary 6.15. *The threefolds $X_{3,3(i)}$ for $i \in \{3, 4, 5, 6\}$ admit a primitive, type III contraction to a threefold Y_i which is smoothable, thus yielding a smooth Calabi-Yau threefold $\tilde{Y}_{3,3(i)}$.*

Theorem 6.16. *For discussed $X = X_{3,3(i)}$ with $i \in \{3, 5, 6\}$ the morphism $\phi_{(|m(2H+E)|)} : X \to Y$ for some $m >> 0$ is a primitive contraction of type III which contracts the ruled surface E to a curve $\tilde{C}$. For $X = X_{3,3(4)}$ the morphism $\phi_{(|m(3H-D+E)|)} : X \to Y$ for some $m >> 0$ serves the same role.*

Tables 9 and 10 summarize the results obtained for complete intersection threefolds.

Remark 6.17. As in the case of a complete intersection of a cubic and a quintic we can analyse the threefold $X_{3,3}$ containing a cone over a degree 9 curve which is the complete intersection of two cubics. Then we have $E = H_X$. In this case we have no excess lines and no double points of $\bar{X}$ lying on the ruled surface E. We can still apply the formulas for $h^{1,1}$ and $h^{1,2}$ to obtain their values for $X_{3,3(3,3)}$. For a smooth threefold $X_{3,3(3,3)}$ we have $\delta = 0$, $h^{1,1} = 2$ and $h^{1,2} = 62$.

6.3. **Calabi-Yau threefolds from sextic threefolds.** Recall that in the case of a sextic in a weighted projective space the curves serving as a basis of a ruled surface need to be contained in $\mathbb{P}^2$ and thus be plane curves. As we also want them to contained in cubic surface this leaves us with only 3 cases of non-singular curves of degrees 1, 2 and 3. Out of these three only the last one has $g \geq 1$ and so we arrive at the following results.

Theorem 6.18. *Threefolds $X_{6(i)}$ constructed in this paper admit*

(1) a type I, a type II, and a type III primitive contractions for $X_{6(3)}$
(2) a type I and two type III primitive contraction for $X_{6(2)}$
(3) two type I primitive contractions and a type III primitive contraction for $X_{6(1)}$.

Corollary 6.19. *The threefold $X_{6(3)}$ admits a primitive, type III contraction to a threefold $Y_{6(3)}$ which is smoothable, thus yielding a smooth Calabi-Yau threefold $\tilde{Y}_{6(3)}$.*

The Table 13 sumarizes the results obtained for sextic threefolds. Note that even though $\tilde{Y}_{6(3)}$ is smooth we do not have the formula to calculate its Hodge numbers as $g(C) = 1$. In the Table 9.2 we present the Hodge numbers of threefolds X_6 obtained by the resolution of singularities of sextic threefolds with triple point and a cone.

7. Appendix A: Possible number of triple points on Calabi-Yau threefolds

In this section we consider the bounds pertaining the number of ordinary triple points on Calabi-Yau threefolds X. We always assume that the threefold in question is to have only ordinary triple points as singularities. Let $\mu_3(X)$ denote the maximal number of ordinary triple points that X can have under this condition. We drop X if the context is clear.

7.1. Complete intersection $X_{2,4} \subset \mathbb{P}^5$.

We consider a threefold being an intersection of a quadric and a quartic fourfold in $\mathbb{P}^5$.

Theorem 7.1. *A complete intersection threefold $X_{2,4} \subset \mathbb{P}^5$ with only ordinary triple points as singularities can contain at most* 10 *of them.*

Proof. We can assume that the equations of X_2 and X_4 are as follows $F_2 = wG_1 + G_2$, $F_4 = wG_3 + G_4$ where G_i are homogeneous polynomials of degree i independent of w and where $O = [0 : 0 : 0 : 0 : 0 : 1]$ is one of the triple points of $X_{2,4}$. We can now perform a projection from the point O onto a $\mathbb{P}^4$ and obtain a quintic threefold Q. Let E denote the exceptional divisor of a blowup of a point O. The projection is given by a linear system of strict transforms of hyperplanes passing through the point O that is $|H-E|$ and $(H-E)^3 = 5$ indeed providing us with a threefold of degree 5. We want to show that Q is normal. Let us assume to the contrary that Q is not normal. Then by [38, p.254] there would exist a normalization $\bar{Q}$ such that the ramification locus D would be two-dimensional on Q. A point P is in D if the line L over it cuts $X_{2,4}$ either in two points outside of O or if it is tangent to $X_{2,4}$ at some point. This means that there is a surface $S \subset X_{2,4}$ that is either projected two-to-one into Q or that lines connecting O and S are tangent to $X_{2,4}$. Thus we see that lines cutting S twice or tangent to it need to be contained both in X_2 and X_4. Indeed, counting with multiplicities in case of the tangency, these lines cut the fourfolds in 3 points (in case of X_2) or a triple point and 2 additional points (in case of X_4). Thus they are contained in $X_{2,4}$ and so it means that the projection from O contracts a threedimensional cone ruled over S which is a contradiciton.

Every triple point O_i different than O on X translates into a triple point P_i on Q thus if $\mu_3(X) = n$ then $\mu_3(Q) = n - 1$. Also note that we can consider $G_1G_4 + G_2G_3$ to be the equation of Q and thus there are 24 double points on Q, namely those for which $G_1 = G_2 = G_3 = G_4 = 0$. By an argument similar to [30, Lemma 2.6] we can consider Varchenko's spectral bound for Q. For $\alpha = 2/5$ the spectrum of the fivefold point has length 155 in the interval $(2/5, 7/5)$ whilst the spectrum of the triple point has length 14 and the spectrum of the double point has length 1 in this interval. Thus from $155 - 24 = 131$ and $10 \cdot 14 = 140$ a quintic threefold containing 24 double points can have at most 9 ordinary triple points as the remaining singularities. As then $n - 1 = 9$ we have that $X_{2,4}$ can contain at most $n = 10$ ordinary triple points. □

Lemma 7.2. *There exits a threefold $X_{2,4}$ containing 7 ordinary triple points as only singularities.*

Proof. We can easily produce a fourfold X_4 admitting 6 ordinary triple points in a general linear position. Indeed, the space of quartic fourfolds is of dimension 125 and a triple point imposes 20 linear conditions. Let X_4 be a quadric fourfold having triple points in $[1:1:1:1:1:1]$ and $[1:0:0:0:0:0], \ldots, [0:0:0:0:1:0]$. Let $\tilde{F}_4(x, y, z, t, u, w)$ be the equation defining X_4. We can write $\tilde{F}_4 = w^3F_1 + w^2F_2 + wF_3 + F_4$ where F_i are homogeneous degree i polnomials independent of w. Now lets take $X_2 = V(\tilde{F}_2)$ with $\tilde{F}_2 = wF_1 + F_2 + F_1H_1$ such that X_2 passes through all the triple points of X_4 with H_1 homogenous linear independent of w. For F_i general enough the complete intersection $X_{2,4}$ will have an ordinary triple point at $[0:0:0:0:0:1]$ and inherit all the ordinary triple points of X_4 producing $X_{2,4}$ with 7 OTPs. We are unaware of $X_{2,4}$ having more than 7 OTP's as the only singularities. □

7.2. **Complete intersection $X_{3,3} \subset \mathbb{P}^5$.** We briefly discuss the complete intersection threefold $X_{3,3}$. This has proven to be more complicated because of the different ways we can obtain triple points on $X_{3,3}$. We descibe here those two ways and also analyse what is the image of the projection from the triple point O in both cases. We do not provide the upper bound yet we construct interesting examples of $X_{3,3}$ admitting $3, 6$ and 9 ordinary triple points.

We consider a degree 9 threefold being an intersection of two degree 3 hypersurfaces in $\mathbb{P}^5$. As usual we assume that $X_{3,3}$ has only ordinary triple points as the only singularities. Let $X_{3,3}$ be $V(F_3, G_3)$ where F_3 and G_3 are equations of degree 3 fourfolds in $\mathbb{P}^5$. Any other cubic fourfold containing $X_{3,3}$ is an element of a pencil $aF_3 + bG_3$ for $a, b \in \mathbb{C}$.

It is important to distinguish two cases. A triple point P on $X_{3,3}$ may be coming from a triple point on some cubic fourfold containing $X_{3,3}$ or there may be not exist such a fourfold. This is the case for example when $F_3 = x^2A_1 + xA_2 + A_3$ and $G_3 = x^2A_1 + x(A_2 + B_1A_1) + B_3$ with A_i, B_i being homogeneous of degree i independent of x. The tangent cone at $O = [1 : 0 : 0 : 0 : 0 : 0]$ of $V(F_3, G_3)$ is generated by A_1 and $A_3 - B_3$ so for general A_1, A_3, B_3 this is an ordinary triple point yet there is no cubic fourfold with triple point at O.

We begin with the case when the triple point O of $X_{3,3}$ is not inherited from a cubic fourfold. We want to discuss the projection of $X_{3,3}$ on $\mathbb{P}^4$ from this triple point $P = [1 : 0 : 0 : 0 : 0 : 0]$.

Lemma 7.3. *Assume that the triple point O of $X_{3,3}$ is not inherited from a cubic fourfold. Let $\tilde{X}_{3,3}$ be the threefold obtained as a resolution of this point. Then $\tilde{X}_{3,3}$ can be projected onto a sexic threefold singular along a degree three surface.*

Proof. We use the equations $F = x^2F_1 + xF_2 + F_3$ and $G = x^2F_1 + x(\alpha F_2 + H_1F_1) + G_3$ for X_3 and Y_3 such that $X_{3,3} = X_3 \cap Y_3$. We blow up O and obain $X_{3,3}$. Projection from the preimage of O is given by a complete linear system of pullbacks of hyperplanes passing through O. Let E denote the exceptional divisor over O. Then this system is $|\tilde{H} - E|$ and $(\tilde{H} - E)^3 = 6$ meaning that the image threefold of the projection is a sextic X_6 in $\mathbb{P}^4$. We can calculate the equation of this sextic to be $F_6 = (F_3 - G_3)^2 + \alpha(F_3 - G_3)F_2H_1 + F_1H_1^2F_3$. This can be seen directly by solving $F = G = 0$ or as the resultant of F and G treated as quadratic polynomials with respect to x. This threefold is singular along a degree three surface S_3 being the vanishing locus of $(F_3 - G_3) = H_1 = 0$. This surface is the image of a the locus of $X_{3,3}$ that is being projected two-to-one to F_6.

Note that S_3 is not the only singular locus of X_6. Let us consider the images of other triple points in $X_{3,3}$. Let P_1 be a triple point and let L be the line spanned by P and P_1. In case there is no other triple point on L we have that the image Q_1 of P_1 is again a triple point. It may happen that the line L is contained in $X_{3,3}$ and that there is another triple point on L. Then the projection contracts this line to a fourfold point on X_6. □

We now discuss with the case when the triple point O on $X_{3,3}$ are inherited from triple points on some cubic fourfolds.

Lemma 7.4. *Assume that in a pencil of cubic fourfolds containing a complete intersection threefold $X_{3,3} \subset \mathbb{P}^5$ with only triple points as singularities there is a fourfold with triple point. Then a threefold $\tilde{X}_{3,3}$ obtained by blowing up one of the triple points of $X_{3,3}$ is a double cover of a cubic threefold Y_3 ramified over a degree 12 surface.*

Proof. Without loss of generality we may assume that $X_{3,3}$ is the complete intersection of a fourfold $X_3 = V(x^2G_1 + xG_2 + G_3)$ and a cone $\bar{Y}_3 = V(F_3)$ where G_i and F_i are homogeneous polynomials of degrees i independent of the variable x. Note that in this case P is exactly the vertex of the cone $\bar{Y}_3$ through which X_3 passes smoothly. A general line in $\mathbb{P}^5$ cuts X_3 in three points. As X_3 passes through P it means that it cuts a general line of the ruling of the cone $\bar{Y}_3$ in 2 other points. As we blow up P and obtain $\tilde{X}_{3,3}$ we can treat it as a double cover of the base of the cone that is $Y_3 \subset \mathbb{P}^4$. We are still to check the ramification locus. This is exactly the vanishing locus of the discriminant $D := V(G_2^2 - 4G_1G_3)$ calculated with respect to x from the equation defining X_3. Intersection of D and Y_3 in $\mathbb{P}^4$ is a degree 12 surface S_{12} which concludes the proof. □

In particular if we have different types of triple points on $X_{3,3}$ (inherited or not) we can choose one of the projections and obtain a double cover of a cubic threefold or a birational map onto a sextic threefold.

Lemma 7.5. *There exists a complete interection threefold $X_{3,3}$ with $3, 6$ and 9 ordinary triple points as only singularities.*

Proof. We obtain threefolds $X_{3,3}$ with only ordinary triple points as singularities by intersecting fourfold cubics containing a triple line. Assume X and Y are fourfold cubics in $\mathbb{P}^4$ such that $X_{3,3} = V(X) \cap V(Y)$. If X admits a triple line and Y is smooth we obtain an $X_{3,3}$ with three colinear triple points. If X and Y both have triple lines without common point then $X_{3,3}$ admits 6 OTP's. Finally if we take $X = V(F) = V(x^3 + y^3 + z^3 + t^3)$ and $Y = V(G) = V(w^3 + u^3 - z^3 - t^3)$ it is easy to verify that $Z = V(F + G)$ is a cubic fourfold containing $X_{3,3}$ and each of X, Y, Z admits a different triple line yielding 9 OTP's on $X_{3,3}$ and finishing the proof. □

We are not aware of any threefold $X_{3,3}$ with more than 9 ordinary triple points.

7.3. **Sextic in** $\mathbb{P}[1:1:1:1:2]$.

Theorem 7.6. *A sextic hypersurface is* $\mathbb{P}[1:1:1:1:2]$ *with only ordinary triple points as singularities can have at most* 10 *of them and this limit is attainable.*

We work in a weighted projective space $\mathbb{P}[1:1:1:1:2]$ with variables x, y, z, t of weights 1 and u being a variable of weight 2. We can thus write the equation of a sextic hypersurface as $F(x, y, z, t, u) = u^3 + u^2 G_2(x, y, z, t) + u G_4(x, y, z, t) + G_6(x, y, z, t)$ where G_i are homogeneous polynomials of degrees i. Note that the term u^3 has to appear in this equation as otherwise the sextic would be passing through the singular point of WPS and inherit its singularity.

Lemma 7.7. *All triple points of a sextic hypersurface* X *in* $\mathbb{P}[1:1:1:1:2]$ *have to lie on a sextic surface being a hyperplane section of* X*.*

Proof. For A to be an (ordinary) triple point of X we need all the second derivatives of F to vanish. In particular we need $\frac{\partial^2 F}{\partial u^2} = 6u + 2G_2(x, y, z, t) = 0$ and thus A has to lie on $V(3u + G_2(x, y, z, t))$ which is isomorphic to $\mathbb{P}^3$. This $\mathbb{P}^3$ cuts X in what is isomorphic to a sextic surface. □

We can perform the change of variables so the hyperplane section of X is $u = 0$. Thus we can consider triple points of X as lying on a sextic surface in $\mathbb{P}^3$ which is the vanishing locus of $G_6(x, y, z, t)$.

Lemma 7.8. *Triple points of a sextic threefold X are also the triple points of its hyperplane section $S_6 = V(G_6)$.*

Proof. Again we take a look at the partial derivatives. As we changed the variables so all triple points lie in $V(u)$ we can work in the chart $x \neq 0$ and, if necessary move to $y \neq 0$. We consider $\frac{\partial F}{\partial \alpha} = u^2 \frac{\partial G_2}{\partial \alpha} + u \frac{\partial G_4}{\partial \alpha} + \frac{\partial G_6}{\partial \alpha}$ and $\frac{\partial^2 F}{\partial \alpha \partial \beta} = u^2 \frac{\partial^2 G_2}{\partial \alpha \partial \beta} + u \frac{\partial^2 G_4}{\partial \alpha \partial \beta} + \frac{\partial^2 G_6}{\partial \alpha \partial \beta}$ for $\alpha, \beta \in \{x, y, z, t\}$ depending on the charts. As these have to vanish at the point A with $u = 0$ we immediately obtain that, slightly abusing notation, A is also at least triple on S_6. To show that A cannot be a point of higher multiplicity on S_6 we look at the local equation of X around A. Assume that A is a point of multiplicity 4 or higher on S_6. Without loss of generality we can write that A is $[1 : 0 : 0 : 0 : 0]$. Then the local equation of S_6 around A is $x^2\tilde{G}_4 + x\tilde{G}_5 + \tilde{G}_6$ and the local equation of X around A begins with $u^3 + u^2H_1 + uH_2$ with H_i being forms of degree i in G_{i+1} effectively preventing it from being an ordinary triple point of X. □

From [11, Proposition 3.2] we know that if S_6 is a normal surface it can have at most 10 (ordinary) triple points. Thus if we want to obtain more that 10 triple points on X we need to assume that S_6 is not normal. From that S_6 needs to have at least one-dimensional singular locus. We being with the case when $\dim(\mathrm{Sing} S_6) = 2$. This is for example the case when S_6 is a triple quadric.

Lemma 7.9. *Sextic threefold in $\mathbb{P}(1 : 1 : 1 : 1 : 2)$ whose triple points lie on a hyperplane section with a two-dimensional singular locus admits a singular curve.*

Proof. We have $X = V(u^3 + u^2G_2 + uG_4 + G_6)$. Note that on a curve $C = V(u) \cap (V(G_4)) \cap (\mathrm{Sing} S_6) \subset X$ all partials of X vanish. This finishes the proof. □

From now on we are only concerned with the case where S_6 is a non normal surface with one-dimensional singular locus.

Proof of 7.6. Let X have the equation $F(x,y,z,t,u) = u^3+u^2G_2(x,y,z,t)+uG_4(x,y,z,t)+G_6(x,y,z,t)$. Let G_6 be the equation of a non-normal sextic surface with at least 11 triple points, denote by Σ the set of these triple points. We modify the argument regarding the polar bound to encompass the case where S_6 is not normal. Let us consider $|S_5|$ - a linear system of polar surfaces of S_6 that is a system of quintic surfaces generated by the partial derivatives of G_6. The base locus of this system is a singular curve of S_6 and a set Σ taken with multiplicity two, thus the general member X_5 of $|S_5|$ will be a quintic containing the singular curve of S_6 and having double points at Σ. Note that the dimension of intersection of S_6 and a general polar surface is 1 as expected and contains the singular curve of S_6. Now, S_6, X_5 and $X_4 = V(G_4)$ should intersect only in a finite number of points. As we know that all these surfaces contain points of Σ as triple or double points we can calculate degree of this intersection to be at least $11 * 3 * 2 * 2 = 132$ which is more than expected 120 and thus there needs to be at least a common curve of intersection for all these surfaces. As this calculation is true for all polar surfaces of S_6 this curve has to be contained in the singular curve of S_6 but then, from calculating the partials, it would be singular on X contradicting the assumption on singular locus of X having only isolated ordinary triple points.

As for the proof of the final statement of the theorem it is enough to use the equation $X = V(u^3 + G_6)$ where G_6 is any of the sextics with 10 OTPs as in [11] or [40]. It is clear that X cannot be singular for $u \neq 0$ and for $u = 0$ only singularities are those on $V(G_6) = S$. Also, X misses the singular point of the WPS as there is a monomial u^3 in its equation. □

Remark 7.10. In [6, Remark 2.2] Cynk calculates the defect δ and the Hodge numbers of an exemplary sextic threefold $X \subset \mathbb{P}(1:1:1:1:2)$ with equation of the form $G = u^3 + G_6$ where $V(G_6)$ is one of the sextic surfaces with 10 triple points as described in [11]. In particular he obtains $\delta = 10, h^{1,1}(X) = 21$ and $h^{1,2}(X) = 3$.

7.4. **Calabi-Yau threefolds with no triple points.** We briefly discuss the complete intersection threefold $X_{2,2,2,2} \subset \mathbb{P}^7$. In a similar spirit we mention two hypersurfaces in weighted projective space that cannot admit triple points.

Proposition 7.11. *A complete intersection threefold $X_{2,2,2,2} \subset \mathbb{P}^7$ cannot have triple points.*

Proof. We cannot obtain an ordinary triple point P as the tangent cone of $X_{2,2,2,2}$ at P needs to be a cone over a smooth cubic surface. This means that there could exist a change of variables such that locally around P the equation of $X_{2,2,2,2}$ would be $F_3 = x = y = z = 0$. As all the sixfolds are of degree 2 this cannot happen. □

Proposition 7.12. *There can be no ordinary triple points on $X_8 \subset \mathbb{P}(1:1:1:1:4)$ or $X_{10} \subset \mathbb{P}(1:1:1:2:5)$.*

Proof. Hypersurfaces we discuss cannot pass through the singular points of the ambient projective spaces. In the case of an octic the singular point of the WPS is of higher multiplicity than we allow. In the case of the degree 10 hypersurface the point of multiplicity 5 is to be missed for the same reasons. We also have that X_{10} cannot pass through the point of multiplicity 2 as then the singular point on X_{10} could not be an ordinary triple point as 3 is not a multiple of 2. From the above the equation of $X_8 \subset \mathbb{P}(1:1:1:1:4)$ ($X_{10} \subset \mathbb{P}(1:1:1:2:5)$) necessarily involves the term with the last variable u in the second power. As we can choose coordinates so that the putative triple point is $O = [1:0:0:0:0]$ we see that in the chart $x = 1$ the equation of X begins with u^2 meaning O cannot be a triple point, a contradiction. □

8. Appendix B: Open problems

It is natural to consider in our constructions more general singularities then triple points.

8.1. **Canonical singularities.** Recall [35, Definition (1.1)] and [35, Definition (2.4)]:

Definition 8.1. A quasi-projective variety X has canonical singularities if it is normal and if the following conditions hold:

(1) for some integer r we have $\omega_X^{[r]}$ locally free;
(2) for some resolution $f : \tilde{X} \to X$, and r as above $f_*\omega_{\tilde{X}}^{\otimes r} = \omega_X^{[r]}$.

Any singularity P of X such that these conditions hold locally in the neighbourhood of P is called a canonical singularity.

Definition 8.2. A Gorenstein point $P \in X$ is rational (elliptic) if for a resolution $f : \tilde{X} \to X$ we have $f_*\omega_{\tilde{X}} = \omega_X$ $(f_*\omega_{\tilde{X}} = \mathfrak{m}_P \cdot \omega_X)$.

Also recall [35, Corollary (2.10)]:

Corollary 8.3. *To a rational Gorenstein* 3*-fold point* $P \in X$ *we can attach a natural number* $k \geq 0$ *such that:*

(1) $k = 0$ *if and only if* P *is a cDV point*
(2) $k \geq 1$ *if the general section* H *through* P *has an elliptic Gorenstein point* $P \in H$ *with invariant* k
(3) if $k \geq 2$ *then* $k = mult_P X$
(4) if $k \geq 3$ *then* $k + 1 = \dim \mathfrak{m}/\mathfrak{m}_P^2$.

Resolution of rational Gorenstein singularities does not change the canonical class of a variety. We are interested in the possibility of constructing Calabi-Yau threefolds containing a cone whose vertex is a such a point and that their resolution would admit a (smoothable) type III contraction.

We begin by discussing the examples of threefolds containing a cone over a curve with vertex being a point that locally looks like the complete intersection of two quadrics. This is an isolated rational Gorenstein point with invariant $k = 4$ (as in [18, 3.3]). In the latter we call it a $(2, 2)$ point for brevity. Below we present our constructions of Calabi-Yau threefolds with a $(2, 2)$ singularity at the vertex of a cone over a curve.

8.2. **Complete intersection threefolds with $(2, 2)$ point at the vertex of a cone.**

8.2.1. *Complete intersection* $X_{2,2,2,2}$. We want to obtain threefold in $\mathbb{P}^7$ obtained as an intersection of four six-dimensional hypersurfaces of degree 2 containing a $(2, 2)$ singularity. To that aim we can assume two quadrics to be cones in $\mathbb{P}^7$. We need the curve C to be contained in 4 quadrics in $\mathbb{P}^6$ and also in two hyperplanes in $\mathbb{P}^6$ making it effectively a curve in $\mathbb{P}^4$ contained in the intersection of four quadrics. Let A_1, B_1 be the equations of the hyperplanes and A_2, B_2, C_2, D_2 the equations of quadrics. Then $X_2 = V(x_7A_1 + A_2)$, $Y_2 = V(x_7B_1 + B_2)$, $Z_2 = V(C_2)$ and $W_2 = V(D_2)$ describe 4 quadrics containing a cone over the given curve in $\mathbb{P}^7$ and so their intersection is a Calabi-Yau threefold with a $(2, 2)$ singularity at the vertex of the cone.

8.2.2. *Complete intersection* $X_{3,3}$. To construct desired threefold we require two quadrics and two cubics in $\mathbb{P}^4$ containing the same curve. Let A_2, B_2 and A_3, B_3 be the equations of quadrics and cubics respectively. Then the complete intersection in $\mathbb{P}^5$ of $X_3 = V(x_5A_2 + A_3)$ and $Y_3 = V(x_5B_2 + B_3)$ is the desired Calabi-Yau threefold containing a cone over a curve with a $(2, 2)$ singularity at the vertex of the cone.

8.2.3. *Complete intersection* $X_{2,4}$. Now we need two quadrics $V(A_2)$, $V(B_2)$ and a quartic $V(B_4)$ containing a given curve in $\mathbb{P}^4$. Then the intersection of $X_2 = V(A_2)$ and $X_4 = V(x_5^2B_2 + B_4)$ in $\mathbb{P}^5$ is the desired threefold. We may also consider a cubic hypersurface

$V(B_3)$ containing the curve and thus obtain $X_4 = V(x_5^2B_2 + x_5B_3 + B_4)$ but it is not necessary for the construction.

8.2.4. *Complete intersection* $X_{2,2,3}$. We may either obtain the singular point as the intersection of two cones in $\mathbb{P}^6$ over quadrics in $\mathbb{P}^5$ intersected with a smooth cubic hypersurface or as an intersection of a cone over quadric and a cubic hypersurface having a double point with a smooth quadric. In the first case we need $V(A_2)$, $V(B_2)$, $V(C_1)$ and $V(C_3)$ in $\mathbb{P}^5$, two quadrics, a cubic and a hyperplane containing a given curve. Then by writing $X_2 = V(A_2)$, $Y_2 = V(B_2)$, $Z_3 = V(x_6^2C_1 + C_3)$ in $\mathbb{P}^6$ we obtain the desired threefold as an intersection $X_2 \cap Y_2 \cap Z_3$. In the second case we need a hyperplane $V(A_1)$, quadrics $V(A_2)$, $V(B_2)$, $V(C_2)$ and a cubic $V(C_3)$ containing a curve in $\mathbb{P}^5$. Then in $\mathbb{P}^6$ we have $X_2 = V(x_6A_1 + A_2)$, $Y_2 = V(B_2)$ and $Z_3 = V(x_6C_2 + C_3)$. Again we intersect $X_2 \cap Y_2 \cap Z_3$ to obtain the desired threefold. We see that the second construction is more restrictive as it requires more quadrics in $\mathbb{P}^5$ to contain the original curve. It happens because in the first case we do not require the cubic hypersurface in $\mathbb{P}^6$ to have an x_6C_2 as a part of its equation which would have evened out the number of restrictions. Thus it is easy to see that any threefold obtainable in the second manner should have its counterpart in the first one - meaning a complete intersection threefold containing a cone over the same curve, with a $(2,2)$ singularity at the vertex.

The formula to calculate the Hodge numbers of constructed threefolds is yet to be obtained as the $(2,2)$ point provides more complications than the ordinary triple point. In particular we do not know how to calculate the defect of such threefold. When we discuss other canonical singularities we arrive at the more general open problem.

Problem 8.4. *Let $P_1, \dots, P_n$ be isolated canonical singularities. What is the formula for the defect of a Calabi-Yau threefold X arising as a resolution of a threefold $\bar{X}$ containing m_i singularities of type P_i for $i \in \{1, \dots n\}$ and $m_i \in \mathbb{Z}^{>0}$?*

8.3. **Calabi-Yau threefolds with singular points.** In spirit of the problems from Appendix A we can also ask the following:

Problem 8.5. *Let $P_1, \dots, P_n$ be isolated canonical singularities and X one of the Calabi-Yau complete intersection threefolds. Assume that X admits only singularities of types P_i. What is the maximal number $\mu_{P_i}(X)$ of singularities P_i that a threefold X can admit? What are the possible n-tuples $(\mu_{P_1}, \dots \mu_{P_n})$ such that X can admit simultaneously μ_{P_i} points of type P_i as the only singularities.*

In particular the still open problem of finding $\mu_3(X_5)$ - the maximal number of ordinary triple points on a quintic threefold - is of this type.

9. Appendix C: Calculations

9.1. **Defect of quintic threefold and sextic threefold with triple and double ordinary points.** Here we present the code for calculating the defect of quintic threfolds in $\mathbb{P}^4$ and sextic threefolds in weighted projective space with triple and double points. Here X denotes the ideal generated by the defining equation F_5 or F_6. The triple point of X is $[1:0:0:0:0:0]$.

```
J=ideal singularLocus X
O=ideal(y,z,t,u)
O3=O^3
JR=J:O^17
JO=J+O3
JJ=intersect(JO,JR)
J6=basis(6,JJ)
```

Let j_6 denote the cardinality of J6. Now the defect is just $\delta = 175 - j_6 - \mu_2 - 11$ for sextic in $\mathbb{P}(1:1:1:1:2)$ as per formula above. Analogous calculations of a quintic threefold differ by calculating

```
J5=basis(5,JJ)
```

instead of J6 at the last step. Then the formula for the defect is $\delta = 101 - j_5 - \mu_2 - 11$.

9.2. **Defect of complete intersection threefolds with triple and double ordinary points.** Let X be the complete intersection threefold $X_{2,4}$ or $X_{3,3}$. The minors of $Jac(X)$ necessarily vanish at the singular points of X. In particular it may happen that one of the rows of $Jac(X)$ vanishes at some singular points of X. To avoid that we use matrix M whose rows are linear combinations of rows of $Jac(X)$ which do not vanish at any singular points of X.

```
O=ideal(x,y,z,t,u)
O3=O^3
JR=J:O^17
JO=J+O3
JM=intersect(M,JO,JR)
M5=super basis(5,M)
J5=super basis(5,JM)
```

Again putting j_5 to denote the number of elements of $J5$ we obtain formulae $\delta = 89 - j_5 - 11 - \mu_2$ for the defect of $X_{2,4}$ and $\delta = 73 - j_5 - 11 - \mu_2$ for the defect of $X_{3,3}$.

Table 1. Intersection of curves and divisors on Calabi-Yau threefolds

	H_X	E	D
l	1	-2	1
t	1	0	1
r	0	1	0
h	0	a	-3
e_i	0	b_i	-1
C_0	0	$a^2 - \Sigma b_i^2$	$-3a + \Sigma b_i$

TABLE 2. Curves at a basis of a cone in $\bar{X}_5$

C	$g(C)$	K_C	a	b_1	b_2	b_3	b_4	b_5	b_6
C_1	0	-2	1	1	1	0	0	0	0
C_2	0	-2	1	1	0	0	0	0	0
TC	0	-2	1	0	0	0	0	0	0
C_3	1	0	3	1	1	1	1	1	1
C_{4A}	1	0	4	2	2	1	1	1	1
C_{4B}	0	-2	4	3	1	1	1	1	1
C_{5A}	2	2	5	2	2	2	2	1	1
C_{5B}	2	2	4	2	1	1	1	1	1
C_6	4	6	7	3	3	3	2	2	2
$C_{2,3}$	3	4	6	2	2	2	2	2	2
C_{7A}	1	0	3	1	1	0	0	0	0
C_{7B}	5	8	7	3	3	2	2	2	2
C_{7C}	4	6	7	4	2	2	2	2	2
C_{8A}	6	10	8	3	3	3	3	3	1
C_{8B}	7	12	7	3	2	2	2	2	2
C_{9A}	10	18	9	4	4	3	3	2	2
C_{9B}	9	16	11	4	4	4	4	4	4
$C_{3,3}$	8	14	9	3	3	3	3	3	3
C_{10A}	12	22	11	4	4	4	4	4	3
C_{10B}	11	20	10	4	4	4	3	3	2
C_{11}	15	28	11	4	4	4	4	3	3
$C_{3,4}$	19	36	12	4	4	4	4	4	4
$C_{3,5}$	31	60	15	5	5	5	5	5	5

Table 3. Divisor L on X_5

				$L_{\|D}$						
X	β	L	L^3	a	b_1	b_2	b_3	b_4	b_5	b_6
X_1	-1	3H-D+E	140	4	2	2	1	1	1	1
X_2	0	2H+E	0	1	1	0	0	0	0	0
X_{TC}	0	2H+E	54	1	0	0	0	0	0	0
X_3	1	H+D+E	8	0	0	0	0	0	0	0
X_{4A}	0	2H+E	64	4	2	2	1	1	1	1
X_{4B}	0	2H+E	60	4	3	1	1	1	1	1
X_{5A}	1	H+D+E	12	2	1	1	1	1	0	0
X_{5B}	1	H+D+E	12	1	1	0	0	0	0	0
X_6	1	H+D+E	0	4	2	2	2	1	1	1
$X_{2,3}$	2	2D+E	16	0	0	0	0	0	0	0
X_{7A}	0	2H+E	82	3	1	1	0	0	0	0
X_{7B}	1	H+D+E	24	4	2	2	1	1	1	1
X_{7C}	1	H+D+E	20	4	3	1	1	1	1	1
X_{8A}	1	H+D+E	28	5	2	2	2	2	2	0
X_{8B}	2	2D+E	0	1	1	0	0	0	0	0
X_{9A}	2	2D+E	4	3	2	2	1	1	0	0
X_{9B}	1	H+D+E	2	5	2	2	2	2	2	2
$X_{3,3}$	3	-H+3D+E	36	0	0	0	0	0	0	0
X_{10A}	2	2D+E	8	5	2	2	2	2	2	1
X_{10B}	2	2D+E	4	4	2	2	2	1	1	0
X_{11}	3	-H+3D+E	0	2	1	1	1	1	0	0
$X_{3,4}$	4	-2H+4D+E	8	0	0	0	0	0	0	0

TABLE 4. Degree of a variety Y_5 and Hodge numbers of a variety $\tilde{Y}_5$

Y	deg(C)	$\deg Y$	$\tilde{Y}$	$h^{1,1}(\tilde{Y})$	$h^{1,2}(\tilde{Y})$
Y_1	1m	$340m^3$	-	-	-
Y_2	4m	$48m^3$	-	-	-
Y_{TC}	1m	$54m^3$	-	-	-
Y_3	3m	$58m^3$	$\tilde{Y}_3$	?	?
Y_{4A}	4m	$64m^3$	$\tilde{Y}_{4A}$	?	?
Y_{4B}	2m	$60m^3$	-	-	-
Y_{5A}	7m	$74m^3$	$\tilde{Y}_{5A}$	2	55
Y_{5B}	7m	$74m^3$	$\tilde{Y}_{5B}$	2	55
Y_6	10m	$84m^3$	$\tilde{Y}_6$	2	52
$Y_{2,3}$	12m	$88m^3$	$\tilde{Y}_{2,3}*$	2	57
Y_{7A}	7m	$98m^3$	-	-	-
Y_{7B}	15m	$82m^3$	$\tilde{Y}_{7B}*$	2	54
Y_{7C}	13m	$94m^3$	$\tilde{Y}_{7C}*$	2	49
Y_{8A}	18m	$108m^3$	$\tilde{Y}_{8A}*$	2	51
Y_{8B}	20m	$112m^3$	$\tilde{Y}_{8B}$	2	56
Y_{9A}	25m	$126m^3$	$\tilde{Y}_{9A}$	2	58
$Y_{9B}*$	23m	$122m^3$	$\tilde{Y}_{9B}*$	2	53
$Y_{3,3}$	27m	$130m^3$	$\tilde{Y}_{3,3}*$	2	63
Y_{10A}	32m	$144m^3$	$\tilde{Y}_{10A}*$	2	65
Y_{10B}	30m	$140m^3$	$\tilde{Y}_{10B}*$	2	60
Y_{11}	39m	$162m^3$	$\tilde{Y}_{11}*$	2	72
$Y_{3,4}$	48m	$184m^3$	$\tilde{Y}_{3,4}$	2	84
$Y_{3,5}$	75m	$250m^3$	$\tilde{Y}_{3,5}*$	1	119

TABLE 5. Curves at a basis of a cone in $\bar{X}_{2,4}$

C	$g(C)$	K_C	a	b_1	b_2	b_3	b_4	b_5	b_6
C_1	0	-2	1	1	1	0	0	0	0
C_2	0	-2	1	1	0	0	0	0	0
TC	0	-2	1	0	0	0	0	0	0
C_4	1	0	4	2	2	1	1	1	1
C_5	2	2	5	2	2	2	2	1	1
$C_{2,3}$	3	4	6	2	2	2	2	2	2

TABLE 6. Curves at a basis of a cone in $\bar{X}_{3,3}$

C	$g(C)$	K_C	a	b_1	b_2	b_3	b_4	b_5	b_6
C_1	0	-2	1	1	1	0	0	0	0
C_2	0	-2	1	1	0	0	0	0	0
TC	0	-2	1	0	0	0	0	0	0
C_3	1	0	3	1	1	1	1	1	1
C_4	1	0	4	2	2	1	1	1	1
C_5	2	2	5	2	2	2	2	1	1
$C_{2,3}$	3	4	6	2	2	2	2	2	2
C_9	8	14	9	3	3	3	3	3	3

TABLE 7. Divisor L on $X_{2,4}$

				$L_{\|D}$						
X	β	L	L^3	a	b_1	b_2	b_3	b_4	b_5	b_6
X_1	-1	3H-D+E	221	4	2	2	1	1	1	1
X_2	0	2H+E	72	1	1	0	0	0	0	0
X_{TC}	0	2H+E	78	1	0	0	0	0	0	0
X_4	0	2H+E	88	4	2	2	1	1	1	1
X_5	1	H+D+E	15	2	1	1	1	1	0	0
X_6	2	2D+E	0	0	0	0	0	0	0	0

TABLE 8. Divisor L on $X_{3,3}$

				$L_{\|D}$						
X	β	L	L^3	a	b_1	b_2	b_3	b_4	b_5	b_6
X_1	-1	3H-D+E	80	4	2	2	1	1	1	1
X_2	0	2H+E	12	1	1	0	0	0	0	0
X_{TC}	0	2H+E	86	1	0	0	0	0	0	0
X_3	1	H+D+E	96	0	0	0	0	0	0	0
X_4	0	2H+E	16	4	2	2	1	1	1	1
X_5	1	H+D+E	0	2	1	1	1	1	0	0
X_6	2	2D+E	0	0	0	0	0	0	0	0

TABLE 9. Degree of a variety $Y_{2,4}$ and Hodge numbers of a variety $\tilde{Y}_{2,4}$

Y	$\deg(C)$	$\deg Y$	$\tilde{Y}$	$h^{1,1}(\tilde{Y})$	$h^{1,2}(\tilde{Y})$
Y_1	1m	$548m^3$	-	-	-
Y_2	4m	$72m^3$	-	-	-
Y_{TC}	1m	$78m^3$	-	-	—
Y_4	4m	$88m^3$	$\tilde{Y}_4$	2	58
Y_5	7m	$98m^3$	$\tilde{Y}_5$	2	58
Y_6	12m	$112m^3$	$\tilde{Y}_{6A}$	2	62

TABLE 10. Degree of a variety $Y_{3,3}$ and Hodge numbers of a variety $\tilde{Y}_{3,3}$

Y	$\deg(C)$	$\deg Y$	$\tilde{Y}$	$h^{1,1}(\tilde{Y})$	$h^{1,2}(\tilde{Y})$
Y_1	1m	$596m^3$	—	-	—
Y_2	4m	$80m^3$	—	-	—
Y_{TC}	1m	$86m^3$	-	-	—
Y_3	1m	$90m^3$	$\tilde{Y}_3$	?	?
Y_4	4m	$96m^3$	$\tilde{Y}_{4A}$	2	46
Y_5	7m	$106m^3$	$\tilde{Y}_{5A}$	2	46
Y_6	12m	$120m^3$	$\tilde{Y}_6$	2	48
Y_9	27m	$162m^3$	$\tilde{Y}_{9*}$	1	70

Table 11. Curves at a basis of a cone in $\bar{X}_6$

C	$g(C)$	K_C	a	b_1	b_2	b_3	b_4	b_5	b_6
C_1	0	-2	1	1	1	0	0	0	0
C_2	0	-2	1	1	0	0	0	0	0
C_3	1	0	3	1	1	1	1	1	1

Table 12. Divisor L on X_6

				$L_{\|D}$						
X	β	L	L^3	a	b_1	b_2	b_3	b_4	b_5	b_6
X_1	-1	3H-D+E	167	4	2	2	1	1	1	1
X_2	0	2H+E	56	1	1	0	0	0	0	0
X_3	1	H+D+E	9	0	0	0	0	0	0	0

Table 13. Degree of a variety Y_6

Y	deg(C)	$\deg Y$
Y_1	1m	$404m^3$
Y_2	4m	$56m^3$
Y_3	3m	$66m^3$

TABLE 14. Hodge numbers of Calabi-Yau threefolds from threefolds with triple points

X	$\mu_3(X)$	δ	$h^{1,1}(X)$	$h^{1,2}(X)$
$X_{2,4}$	7	16	24	94
$X_{3,3}$	3	6	10	68
$X_{3,3}$	6	12	16	74
$X_{3,3}$	9	26	36	88
$X_{6(1)}$	1	1	3	77
$X_{6(2)}$	1	1	3	65
$X_{6(3)}$	1	1	3	57

TABLE 15. Bound of number of ordinary triple points on Calabi-Yau threefolds

X	$\mu_3(X)$
$X_5 \subset \mathbb{P}^4$	10 or 11
$X_{3,3} \subset \mathbb{P}^5$	$9 \leq$
$X_{2,4} \subset \mathbb{P}^5$	$7 \leq \cdots \leq 10$
$X_{2,2,2,2} \subset \mathbb{P}^7$	0
$X_6 \subset \mathbb{P}(1:1:1:1:2)$	10
$X_8 \subset \mathbb{P}(1:1:1:1:4)$	0
$X_{10} \subset \mathbb{P}(1:1:1:2:5)$	0

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