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Report concerning the awarding of the Habilitation degree to Dr. Piotr Pokora

April 11, 2023

Dear Sir/Madam,

I have been asked to give my opinion concerning the possible awarding the habilitation degree to Dr. Piotr Pokora in the field of natural sciences, in the discipline of mathematics, which will be conducted at the University of Lodz. Specifically, I have been asked for my opinion as to whether Dr. Pokora has scientific achievements constituting a significant contribution to the development of the discipline.

Dr. Pokora received his Ph.D. in 2015 from Uniwersytet Pedagogiczny. According to MathSciNet, he has 33 publications, which appeared in good journals. Specifically, four of his papers appeared in 2022, four in 2021, three in 2020 and four in 2019. This is certainly a very strong output in recent years, and I see no indication that he will slow down.

Furthermore, I was very impressed by his description on page 29 of his Autoreferat ENG of his scientific activity with different international researchers and groups. Such extensive training with so many established researchers and groups is very unusual, and his dramatic and successful record of publications certainly reflects the benefit that he derived from this wonderful research collaboration. In a similar way, his record of scholarships and awards (see page 32 of the Autoreferat ENG), and his long list of talks given, show the positive influence of this training, as well as reflecting his ability and successful contributions to mathematics.

So let me now discuss some of his scientific achievements. As I understand it, I am asked to comment specifically on the seven papers listed on the first page of the document *Wykaz osiągnięć ENG_dr Piotr Pokora*.

Dr. Pokora summarizes his Habilitation work as being concerned with "singular curves in algebraic surfaces – algebra, geometry and combinatorics." This is a fair summary, and much of this work involves configurations of curves on surfaces (including $\mathbb{P}^2$). But this work is attractive to researchers over a broad range of areas, including combinatorics and algebraic geometry. He clarifies that his work has focused largely on the bounded negativity conjecture, constructions of very singular curves in algebraic surfaces, especially in the complex projective plane, and variations on Terao's freeness conjecture and the notion of free and nearly-free reduced plane curves.

The first part of Dr. Pokora's work concerns the so-called *bounded negativity conjecture*. This plays a central role in [Hab1], [Hab2], [Hab3] and [Hab4]. This says, roughly, that given a projective surface X there exists a bound $b(X)$ depending on X so that the self-intersection of every curve on X is at least $-b(X)$ (at least in characteristic zero). Much of the modern work on this topic is due to Brian Harbourne, and Dr. Pokora describes some of Harbourne's work and how his work builds off it. His main work on this topic is contained in the first of his listed papers, [Hab1], which appeared in 2017 in the Journal für reine und angewandte Mathematik (which is a top journal) and was written jointly with Th. Bauer and D. Schmitz. This paper described the connections

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between bounded negativity and so-called Zariski decompositions. These are fundamental topics in the theory of algebraic surfaces. Their first main result is that bounded negativity is equivalent to the boundedness of Zariski denominators. It turns out that there are technical invariants associated to X regarding negativity ($b(X)$) and Zariski decompositions ($d(X)$), and other results from this paper give very strong connections between these.

The Autoreferat written by Dr. Pokora also gives results from other papers of his that are strongly connected to [Hab1]. This includes work with several other authors, including Harbourne. One such result that strikes me as very interesting is a connection between Zariski decomposition and the famous SHGH Conjecture ("SHGH" stands for "Segre-Harbourne-Gimigliano-Hirschowitz"). This description begins on page 11 of the Autoreferat.

Related to bounded negativity are so-called Harbourne indices, which we will not define here. All of this work is very technical, very deep and very central to current research in the area. The paper [Hab4] deals with the Bounded Negativity Conjecture for rational surfaces. This shows that the case where the surface is $\mathbb{P}^2$ is important. It is still open, but much has been done and Dr. Pokora is an active participant in this research. One of the main results of [Hab4] (Theorem B) gives a family of reduced curves in $\mathbb{P}^2$ with ordinary singularities and Harbourne indices that are "very negative." Dr. Pokora's work is also connected to work of Hirzebruch, particularly his bound on multiple points, that we will describe below.

Indeed, the simplest surface is the projective plane, and this simplest plane curve is a line. A natural question is to study the combinatorics of unions of lines in the plane, so-called *line arrangements*. Here the field can also play a role. A second subject in which Dr. Pokora has made quite significant contributions in the area of line arrangements in $\mathbb{P}^2$. There is a lot to describe here. It is not unrelated to the first subject, bounded negativity, but it should be viewed as being almost completely separate.

Line arrangements are unions of lines in the projective plane. There are a lot of questions that one can study about such objects, and Dr. Pokora has delved into many of them. One such question is the role of the field over which one is studying. For example, if the field is $\mathbb{Z}_2$ then the plane contains only 7 elements and 7 lines, so the kinds of questions that can be asked are somewhat limited. On the other hand, over the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$, the theory is rich and the open problems are challenging.

The paper [Hab5] is a survey on Hirzebruch inequalties, but it gives a beautiful exposition of the open problems and the results on the combinatorial side of line arrangements. It gives a good and broad overview of these questions. I really like this.

Let t_r denote the number of r-fold intersection points of the lines (i.e. the number of points through which exactly r points pass). It is an open problem to know exactly which combinations of values of t_r can occur for different r. For example one has the formula of Hirschowitz (Theorem 1.5 in the Autoreferat) that if $\mathcal{L}$ is an arrangement of $d \geq 6$ lines such that $t_d = t_{d-1} = 0$ then

$$t_2 + t_3 \geq d + \sum_{r \geq 5} (r-4) t_r.$$

Also

$$2t_2 + t_3 \geq 3 + d + \sum_{r\geq 5}(r-4)t_r$$

(Hirzebruch-Sakai) with equality if and only if $\mathcal{L}$ is the dual Hesse arrangement (Urzúa). Another interesting inequality is

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{r\geq 5}(2r-9)t_r$$

(proved in the literature but discussed in [Hab5]). Dr. Pokora has derived other inequalities, for example in his paper *Hirzebruch type inequalities and plane curve configurations*, which appeared in the International Journal of Mathematics.

A natural question is when some of these bounds are sharp. One answer, by Uzúa, requires the complex numbers. In work with Bokowski, Dr. Pokora showed that over the reals there is no such arrangement! This is very nice as well.

A related question is to try to find line arrangements which come as close as possible to the theoretical limits (the so-called extreme problems). This is a constructive problem, very different from the theoretical identification of limits. A dual version asks about how how large the possible collinearities are for finite sets of points. Dr. Pokora gives some results and examples along these lines also in [Hab5]. Again, this is a very useful paper.

A very important area of study involves the Jacobian ideal of a hypersurface in $\mathbb{P}^n$. That is, if the hypersurface is defined by a homogeneous polynomial F, then the Jacobian ideal $J(F)$ is the ideal generated by all the first partial derivatives of F. One is interested in fundamental properties of $J(F)$, and of the scheme that it defines. The hypersurface is called *free* if the quotient $R/J(F)$ is Cohen-Macaulay. Equivalently, we ask that the module of syzygies of $J(F)$ be a free module. In the situation of hyperplane arrangements, a famous conjecture of Terao (still open) says that for hyperplane arrangementes, freeness is a combinatorial invariant of the intersection lattice of the arrangement.

One way that F can fail to be free is if $J(F)$ is not saturated. For arrangements in $\mathbb{P}^n$, $n \geq 3$, this is not enough: $J(F)$ can already be saturated and still have that F is not free because the syzygy module is not free. (Technically: $R/J(F)$ has depth ≥ 1 but is not Cohen-Macaulay.) But for arrangements in $\mathbb{P}^2$, freeness is equivalent to $J(F)$ being saturated. We will denote by $J(F)^{sat}$ the saturation of $J(F)$.

In the literature authors have tried to weaken the notion of freeness. Because of the above-mentioned fact about $\mathbb{P}^2$, many authors have focused on curves in the plane, and naturally Dr. Pokora has had a lot to say in this setting. So let us return to $\mathbb{P}^2$. A tool that one finds in the literature is that if F defines a curve in $\mathbb{P}^2$ then one defines the Jacobian module $N(F) = J(F)^{sat}/J(F)$, and the above fact just says that F is free if and only if $N(F) = 0$. Setting $n(F)_j = \dim N(F)_j$ we define

$$\sigma(F) = \min\{j \mid n(F)_j \neq 0\} \quad \text{and} \quad \nu(F) = \max_j\{n(F)_j\}.$$

Then we say that the curve F is *nearly free* if $\nu(F) = 1$. (Notice that this is not the same as saying that there is only one degree in which $J(F)$ fails to be saturated, and that failure misses by the least possible amount. It says that in *each* degree, the failure is either 0 or 1.)

The next step in studying arrangements in $\mathbb{P}^2$ is to leave the realm of *line* arrangements and allow some of the lines to become conics, and to study so-called conic-line arrangements. Dr. Pokora's paper with Alexandru Dimca, [Hab6], gives nice results in this area. They study the situation where the union has only nodes, tacnodes and ordinary triple points as singularities, and seek results related to the famous Terao conjecture mentioned above. Let me briefly describe this paper.

As the authors say, this topic is of interest to both algebraic geometers and to combinatorists, and they have tried to frame their work in both settings. We recall that Terao's conjecture (for line arrangements) says that given two line arrangements with the same combinatorics (basically counting the singular points in the right way), one is free if and only if the other is free. This conjecture is not true if one replaces line arrangements by conic-line arrangements. Indeed, a result of Schenck and Tohaneanu gives a pair of conic-line arrangements for which the statement of Terao's conjecture fails.

[Hab6] seeks to understand the situation better by seeing if Terao's conjecture might hold if one tries to control the singularities. (The possible singularities become more complicated when you allow components that are conics.) It is also necessary to clarify what one means by "the same combinatorics," and the authors give a notion of "weak combinatorics" (inspired by a paper of Marchesi and Vallés). They also formulate a "Numerical Terao's Conjecture" stating that if your arrangements have only nodes, tacnodes and ordinary triple points as singularities, and given two conic-line arrangements with the same weak combinatorics, then one is free if and only if the other is free.

Corollary 5.10 in [Hab6] is a consequence of their Theorem 5.7, which is a complete classification of freeness for conic-line arrangements with only nodes, tacnodes and ordinary triple points as singularities. This is a really stunning result, in my opinion. I think that this paper represents quite important work in the area pf plane arrangements.

Also on the topic of conic-line arrangements, I turn to [Hab7], co-written by Dr. Pokora with T. Szemberg. This paper is unrelated to [Hab6] and does not mention Terao at all. It is more related to the earlier Habilitation achievement I discussed above. The authors begin with the assumption that they have a union of plane curves all of degree 1 or 2, and that this union has only ordinary singularities. (Recall that [Hab6] allowed tacnodes as well, but restricted the multiplicity of each singular point.)

The paper starts by recalling a certain conic-line arrangement that the authors call the *Chilean arrangement*, discovered by Dolgachev, Laface, Person and Urzúa, as well as a related arrangement that they call the *extended Chilean* arrangement. They carefully analyze the combinatorics and the freeness of these arrangements. They extend these ideas, giving what they call a de Bruijn-Erdős type statement. This result gives a simple lower bound for the number of intersection points of a conic-line arrangement with only ordinary singularities. This leads to the main result of this paper (Theorem 4.2): given a conic-line arrangement with only ordinary singularities, let $d \geq 6$ be the number of lines, $k \geq 2$ the number of conics, and t_i the number of i-fold points. Assume $t_{d+k} = 0$

and assume that there exists a subarrangement consisting of six lines intersecting only with double and triple points. Then they show

$$8k + t_2 + 3t_3 + t_4 \geq k + \sum_{r \geq 5} (2r - 9)t_r.$$

The proof of this striking result uses very heavy machinery from algebraic geometry, including $\mathbb{Q}$-divisors, an abelian cover of $\mathbb{P}^2_{\mathbb{C}}$ ramified over the arrangement of order 2, minimal desingularizations, Chern numbers, Kodaira dimension, and so on. It's quite a *tour de force*. As if this were not enough, they then apply these results and constructions to give a result on local negativity, returning full circle to the Bounded Negativity Conjecture with which this report started! Finally they say that they "reopen the so-called geography problem of log surfaces associated with conic-line arrangements," and they study some extremal conic-line arrangements from the viewpoint of log-Chern slopes. Again, this is a powerful paper that gives important results both for combinatorists and for algebraic geometers. It's not often that you see a paper that can make this sort of claim about its breadth.

I hope that in the above report one can see something of the depth and breadth of Dr. Pokora's work. His contributions have been important not only in algebraic geometry but also in combinatorics. He has worked with many very well-known collaborators and has a strong international reputation. In my opinion his scientific achievements constitute a significant contribution to the development of both of the fields I mentioned, and he deserves that his habilitation application be accepted.

Sincerely yours,

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