Review on the scientific work of Dr. P. Pokora, candidate to Habilitation

1 Overall presentation of the research topics of the Candidate

The research of Dr. P. Pokora is devoted to the theory of curves in algebraic surfaces and is particularly focused on three important topics: (i) the bounded negativity conjecture (BNC), (ii) the construction of very singular curves in algebraic surfaces, especially in the complex projective plane, (iii) the Terao's freeness conjecture (TFC). These three topics are variously intertwined.

The BNC is a classical, widely open, very important and extremely difficult problem in algebraic geometry. The question is whether, given any smooth, irreducible, projective surface X over an algebraically closed field of characteristic zero, there exists a (positive) constant $b(X)$ depending on X such that for every irreducible curve C on S one has $C^2 \geq -b(X)$. The conjecture is false in positive characteristic.

An open related question is whether verifying the BNC is a birational invariant or not. In order to analyse this last question an invariant of curves on a surface (the Harbourne index) and of a surface (the global Harbourne index) have been introduced. The behaviour of these invariants is an object of active research in the field.

The search for very singular curves on a surface, especially for reducible plane curves with components of given degrees and assigned type of singularities, is also a very classical subject in algebraic geometry, that has important relationships with combinatorics and widely open problems. One of the main results in this field is a beautiful combinatorial inequality by Hirzebruch which applies to configurations of lines in the complex projective plane, whose only known proof is via the theory of Kummer covers of surfaces.

Finally, the TFC is at the crossroad of various disciplines: algebra, combinatorics, geometry. A reduced plane curve C with equation $f = 0$ is said to be *free* if it is free the module of all derivations that annihilate f. The TFC predicts that if two line arrangements have some identical combinatorial feature, then they are either both free or both not free. The conjecture is known to hold only for configuration of at most 14 lines. A notion of *nearly–freeness* has been given for reduced plane curves. It is believed that possible counterexamples to TFC could be pairs of line arrangements that enjoy the hypothesis of the conjecture and one is free and the other only nearly–free. For this reason free and nearly-free curves gained a lot of attention of researchers working on plane curves.

Next I will briefly describe the very interesting achievements of the candidate for the Habilitation in the above three research fields.

2 Results on the BNC

Let X be a smooth projective surface and D a pseudo-effective divisor on X. The *Zariski decomposition* of D is as follows:

$$D = P + N$$

where P, N are $\mathbb{Q}$–divisors with: (i) P nef, (ii) N effective and negative definite intersection matrix if non–zero, (iii) $P \cdot C = 0$ for every irreducible component C of N.

Since P, N are $\mathbb{Q}$–divisors, in their expression occur denominators. A natural question is the following: does there exist a positive integer $d(X)$ such that for every pseudo-effective integral divisor D the denominators in the Zariski decomposition of D are bounded from above by $d(X)$? If such a bound $d(X)$ exists, then one says that X has *bounded Zariski denominators* (BZD).

The main contribution of the Candidate on the BNC has been the proof of a close relationship between the BNC and the property of having BZD. In the paper [Hab1] (I follow the list of papers provided by the Candidate in his "Habilitation achievement"), he proves that a surface X verifies the BNC if and only if it has BZD. Actually, more than this, he provides explicit bounds between the integers $d(X)$ and $b(X)$. Although this does not solve the BNC, I find this result extremely interesting, since it gives a concrete grasp on the difficulty of the problem and may be a way of attacking it, trying to prove the BZD property.

3 Results on singular curves

I already mentioned the Hirzebruch inequality in §1. This inequality plays a fundamental role in the combinatorics of configurations of lines in the complex projective plane. One of the achievements of the Candidate has been to prove a significant extension of Hirzebruch inequality to d-configurations, that are unions of curves in the complex projective plane, all of the same degree d, pairwise intersecting transversely and not all passing through the same point. The Candidate proves in [Hab2] that for such a configuration with $d \geq 3$ and at least four curves, one has

$$\left(\frac{7}{2}d^2 - \frac{9}{2}d\right) + t_2 + t_3 \geq \sum_{r\geq 5}(r-4)t_r,$$

where for all i, t_i is the number of points where i of the curves of the configuration pass. The result is obtained by a careful analysis of abelian covers of the plane branched over the curves of the configuration, and then applying the Bogomolov-Miyaoka-Yau inequality to such abelian covers. This result has as very interesting corollary a significant lower bound on the so called *degree d-global Harbourne index* which is bounded below by

$$-4 - \frac{5}{2}d^2 + \frac{9}{2}d.$$

In [Hab3] the Candidate extends the above Hirzebruch type of inequality to configurations of curves on smooth complex projective surfaces with numerically trivial canonical class (an importante class of surfaces in Enriques–Kodaira classification). This inequality implies a very interesting lower bound on a type of global Harbourne index for rational curve arrangements having only ordinary singularities on K3 and Enriques surfaces.

In the deep and quite technical paper [Hab4], the Candidate deals with the following question related to the one whether bounded negativity is or not a birational invariant: let X be a smooth complex projective surface, and assume that $b(X) \in \mathbb{Z}$ is a positive integer such that for every irreducible and reduced curve C on X one has $C^2 \geq -b(X)$; let $n \geq 1$ be an integer: is there a positive integer $b(X, n)$ such that for every morphism $\pi : X_\pi \to X$ which is the composition of n point blow-ups, and every irreducible curve C on X_π, one has $C^2 \geq -b(X, n)$? An affirmative answer to this question would imply that any rational surface verifies the BNC, something that is still unknown.

Given any surface X and a positive integer n, one defines

$$h(X, n) = \inf_{X_\pi \to X,\ \text{n blown up points}} \left\{ \inf_{C \subset X_\pi,\ \text{reduced}} \frac{C^2}{n} \right\}$$

so that so that $b(X, n)$ exists if and only if $h(X, n)$ is finite. One of the main results in [Hab4] is the following: set $h = \inf_n h(\mathbb{P}^2, n)$; then for every morphism $X \to \mathbb{P}^2$ which is an n point blow–up, and for every reduced curve C in X, one has $C^2 > hn$. So the above question has an affirmative answer if $h = \inf_n h(\mathbb{P}^2, n)$ is finite.

A second result in [Hab4] is the quite delicate construction of examples of reduced curves C in the complex projective plane with ordinary singularities and Harbourne indices arbitrarily close to $-25/7$. These are the most negative known examples of reduced plane curves with only ordinary singularities. These examples are constructed by starting with a known arrangement of 45 lines due to Wiman, and taking its pull–back via the coverings

$$[x : y : z] \in \mathbb{P}^2 \to [x^k : y^k : z^k] \in \mathbb{P}^2$$

for any integer $k \geq 2$.

The paper [Hab5] is both a survey on Hirzebruch-type inequalities and applications to the search of interesting point–line configurations in the plane, and a research article. The original results concern interesting further extensions of the Hirzebruch inequality to d-configurations of plane curves and to configurations of lines and conics. Further extensions of the Hirzebruch inequality to configurations of lines and conics can be found in the paper [Hab6].

The paper [Hab7] is devoted to a systematic study of conic-line arrangements in the complex projective plane. The candidate proves a couple of interesting equalities of de Bruijn–Erdös type and of Hirzebruch type for certain classes of conic-line arrangements having ordinary singularities. As applications the candidate finds, for instance, the lower bound -4.5 for the Harbourne index of conic-line arrangements in the complex projective plane, and bounds on the log-Chern slopes of open surfaces associated with conic-line arrangements in the plane.

4 Results on the TFC

Regarding the TFC, the Candidate has considered configurations of lines and conics, rather than only configurations of lines. This of course raises the level of difficulty of the problem, but also opens up a wide range of possibility for interesting results in this field.

Here let me mention again the paper [Hab6], in which there is a very interesting classification result for all free arrangements of $d \geq 1$ lines and $k \geq 1$ smooth conics having only nodes, tacnodes, and ordinary triple points as singularities. The very elegant result is that these arrangements are: (i) a smooth conic and a tangent line, (ii) a smooth conic and two tangent lines, (iii) a smooth conic inscribed in a triangle and a smooth conic circumscribed in a triangle, (iv) a triangle, a smooth conic inscribed in the triangle, and another smooth conic circumscribed to the triangle.

5 Conclusions

The research of Dr. P. Pokora concerns the theory of curves in complex algebraic surfaces and particularly focuses on three important topics: (i) the bounded negativity conjecture, (ii) the construction of very singular curves in algebraic surfaces, especially in complex projective plane, (iii) the Terao's freeness conjecture. These are important and very difficult problems in algebraic geometry, in which Pokora obtained extremely good and meaningful results, to the extent that he is now considered to be a world specialist in these fields. His work has been abundant and constant in time. The mathematical culture of Pokora appears to be extremely solid. No doubt that Pokora's scientific achievements constitute an important and significant contribution to the development of important aspects of algebraic geometry. As I can see from his CV, he has also worked out an intense and very qualified didactic activity and of popularization of science, as well as advising and organisational activity. Therefore in my opinion Pokora certainly deserves the recognition of the Habilitation.

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